A UNIFIED APPROACH TO THE THEORY OF NORMED STRUCTURES - PART I: THE SINGLE-SORTED CASE

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Abstract. We introduce the concept of a prenormed model of a particular kind of finitary single-sorted first-order theories, interpreted over a category with finite products. These are referred to as prealgebraic theories, for the fact that their signature comprises, together with arbitrary function symbols (of finite ariety), only relation symbols whose interpretation, in any possible model, is a reflexive and transitive binary relation, namely a preorder. The result is an abstract approach to the very concept of norm and, consequently, to the theory of normed structures.

1. Introduction

There is no doubt that norms, along with diverse analog concepts such as valuations and seminorms, occupy a central place in mathematics, not only in relation to metric spaces and the notion of distance, but also in their own right, as for instance in the theory of Banach spaces [10], valuated rings [6] and normed groups [2] (differently from other authors, we use here the term "valuation" with the meaning of "absolute value"). This article is, in fact, intended as a further proof, if ever necessary, of their centrality: It is one-half of a two-part work in a series of papers devoted to norms and normed structures and culminating with the proof that the tribe of normed structures includes metric spaces among its individual representatives - which is something completely opposed to the common feeling for normed structures have been always regarded as a sort of special metric spaces, and not viceversa. Since the "many-sorted case" is an essentially technical complication of the "one-sorted case" and adds no significant insights to the theory (at least in its basic aspects), we will concentrate here on the latter and consider the former only in the second part. The long-term goal, as well as our original motivation, is the development of a hybrid framework, partly analytical and partly algebraic, where to carry out computations relevant to the a priori convergence theory of approximation schemes in numerical analysis, with a special focus on reduced basis methods [7] (a standard technique used by several authors in applied mathematics to provide effective solutions of numerical problems depending on a large number of parameters): The link is the spectral theory of linear operators and Banach algebras [14], but we are not really going to dig into this in the paper.

Beyond our personal goals, in fact, one stronger motivation for developing a unified theory of normed structures is obviously that, going one rung up the ladder of abstraction, issues, theorems or constructions commonly encountered in this or that particular context could be addressed, proved or accomplished once and for all, without any need to umpteenthly replicate them in every each case as if they were totally independent from each other. On another other,

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from a more philosophical perspective, a higher level of abstraction could suggest, at some point, new directions of research or provide fresh insights for a deeper comprehension of the ontology of familiar concepts. E.g., by separating what is essential from what is accidental in our basic definitions, or by smoothing out, from the natural perspective of categories, a certain degree of arbitrariness that may be inherent to the traditional view.

In this respect, one of the main achievements here is the introduction of an abstract notion of norm for models of a special kind of finitary first-order theories interpreted over a category $\mathbf{K}$ with (all) finite products (see Appendix A). Such theories will be called subalgebraic, due to the fact that their signature includes, together with arbitrary function symbols, only relation symbols whose interpretation is always a partial order (algebraic structures are covered as a special case). The corresponding models will be referred to as $\mathbf{K}$-models, to stress the role of $\mathbf{K}$ in the picture. In particular, it will be proved that all the $\mathbf{K}$-models of a given collection of prealgebraic theories form themselves a category whose morphisms can be eventually understood as "norms". This is used, in turn, to build up another category (over a fixed $\mathbf{K}$-model), whose objects are ultimately an abstraction of normed spaces and whose morphisms are, in a generalized sense, "short maps" between them.

Upon these premises, we show by a number of examples how to recover down-to-earth constructions of common use in the everyday practice, for a variety of applications ranging from functional analysis to linear algebra and number theory, such as normed groups and valuated rings (Part I), normed spaces and normed algebras (Part II), and variants, generalizations or specializations thereof including seminormed semigroups, pseudo-semivaluated fields, normed modules over valuated rings, etc. In this respect, we will face, in the next pages, the following (somewhat vague) questions: What is abstractly a norm? And what are the essential features that one should retain in order to give a purely algebraic definition of a norm, to the degree that normed structures can be ultimately identified with the objects of an appropriate category and norms with the arrows between these objects? The answers provided in the sequel are certainly far from being exhaustive and definite, but we are confident that the subject may be worth the effort and our hope is that it can attract the interest of other researchers in the field, for the benefits are potentially great: push on the use of categorical methods in analysis and analytical methods in category theory.

To the best of our knowledge, the only previous contribution in this line dates back to the 2008 Ph.D. thesis of G.S.H. Cruttwell [8, Ch. 3]. There, partially based on work by M. Grandis [9], the author gives a categorical abstraction of normed groups by regarding a group norm as a (lax) monoidal functor from a compact closed category $\mathbf{D}$ to a monoidal category $\mathbf{M}$. Then, he considers the category $\mathbf{AbNorm}$ with objects given by Abelian normed groups and arrows by group homomorphisms which are also weakly contractive maps. Lastly, he defines a normed (unital) ring $\mathcal{R}$ as a one-object category enriched over $\mathbf{AbNorm}$ and a normed module over $\mathcal{R}$ as an $\mathbf{AbNorm}$-functor $\mathcal{R} \to \mathbf{AbNorm}$ (having once recognized that $\mathbf{AbNorm}$ is a category enriched over itself). Our approach is substantially different (despite of a few points in common). We do not focus on a restricted class of familiar normed structures to describe them from the general perspectives of categories. Rather, we combine the language of categories with that of model theory to invent a general notion of "norm", which applies especially to any arbitrary model of any algebraic theory, to the degree that normed groups, normed rings and normed modules result, among the others, as an instance of a general concept of normed structure.

Many ideas in this paper have been influenced by the prominent work of R. Lowen on approach spaces [13] and F.W. Lawvere on algebraic theories [12] and generalized metric spaces [11]. From

a categorical point of view, extended pseudometric spaces and extended pseudoquasimetric spaces, along with their corresponding Lipschitz maps, have the best properties that one can actually expect from a category of metric spaces: It is possible, within them, to form quotients and take arbitrary products and coproducts. Dropping the attribute “extended” implies that, in general, only finite products and coproducts will exist, while curtailing the prefix “pseudo” affects the existence of quotients. Moving from these considerations, it seems quite reasonable, in search of a “good” answer to questions concerning the “real nature” of norms, to focus first on the weaker notions of seminorm and pseudoseminorm. This leads to one more basic insight, which has been central in this research and can be roughly outlined as follows.

Loosely speaking, a homomorphism of two algebraic structures of the same type, as described in the language of model theory, is a function between the underlying sets with the property of “preserving the operations”. Then, one observes that, with a little effort of imagination, a seminorm, say, on a real vector space exhibits almost the same behaviour:

(i) Its codomain is a special “reference structure”. In the toy case that we are considering, this is the set of non-negative real numbers, herein denoted by $\mathbb{R}_0^+$, together with its standard structure of totally ordered semiring (in this paper, a semiring means a ring without additive inverses and we do not intentionally regard $\mathbb{R}_0^+$ as an ordered semifield).

(ii) It preserves the additive identity (a nullary operation). This has always been something subtle (and, hence, interesting) to our eyes: In the final analysis, one is basically requiring a seminorm to map a distinguished element a in the domain to a distinguished element b in the codomain, in a context where a and b play the same (algebraic) role, but still in a match lining up essentially different teams (both of them are identities, but in structures marked by significant differences).

(iii) It relates a sum (of vectors) to a sum (of scalars) by means of an inequality.

(iv) It equates the product of a scalar by a vector to a product of two scalars, which is informally the same as saying that it preserves the products, except that the one product and the other have very little in common, at least at a first glance.

That said, the next step is to give emphasis to something absolutely obvious, i.e., that equalities and inequalities, appearing in such a fundamental way in the (classical) definition of seminorms, have in common the property of being orders. Some of them are partial, as for the equality relation, while others are total, like in the case of the standard order on the set of real numbers, but they all are orders, i.e., reflexive, antisymmetric and transitive binary relations. And it is just by using orders and relaxing equalities to inequalities that we can manage to relate structures of different types and “let them play a good game.”

This intuition is strengthened by the inspection of other similar constructions encountered in various fields of the mathematical landscape. E.g., a group seminorm can be abstractly defined, based on common terminology and notation from model theory (cf. Remarks 2, 3 and 6), as a function $\|\cdot\|$ from a group $(G;+,-,0_G)$ to an ordered monoid $(M;+,0_M;\leq_M)$ such that $\|a+b\| \leq_M \|a\|+\|b\|$ for all $a,b \in G$ and $\|0_G\| = 0_M$, and this is, indeed, called a group norm if it is symmetric (with respect to the unary operation of negation) and $\|a\| = 0_M$ for some $a \in G$ if and only if $a = 0_G$. Likewise, an absolute value is defined, in the context of ring and field theory, as a function $|\cdot|$ from a domain $(D;+,\cdot,-,0_D)$ to an ordered ring $(R;+,\cdot,-,0_R;\leq_R)$ such that $|\cdot|$ is a group seminorm from the (Abelian) group $(D;+,-,0_D)$ to the (Abelian) ordered monoid $(R_0^+;+,0_R;\leq_R)$ such that $|a\cdot b| = |a|\cdot|b|$ for all $a,b \in D$, where $R_0^+ := \{a \in R : 0_R \leq_R a\}$. Thus, it is naively apparent the existence of a common pattern among these definitions, and the primary goal of the paper is, indeed, to give it an explicit formal description.

1.1. **Basic notation and terminology.** We set our foundations in the Neumann-Bernays-Gödel axiomatic class theory (NBG), as presented in [15, Ch. IV]. We use $\mathbb{N}$ for the non-negative integers and $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ according to their standard meaning. Unless differently stated, each of these sets will be endowed with its ordinary order and operations.

If X, Y are classes, $D \subseteq X$ and $f \subseteq D \times Y$ is such that, for every $x \in D$, there exists only one $y \in Y$ such that $(x, y) \in f$, one says that f is a [total] function (or map, mapping, or similia) $D \to Y$, but also that f is a partial function from X to Y. In this case, D, X and Y are called, each in turn, the domain, the source and the target of f. In particular, we write $\mathrm{dom}(f)$ for D and use the notation $f : X \nrightarrow Y$ (an arrow with a vertical stroke) for a partial function f from X to Y. Note that, formally, a partial map from X to Y is an ordered triple (X, Y, f) for which f is a function $D \to Y$ for some $D \subseteq X$. Yet, we will often identify (X, Y, f) with f when it is convenient to do that and it is clear from the context which classes must be used as source and target. Lastly, if $S \subseteq X$ and g is a function $X \to Y$, then we denote by $g_{|S}$, as is customary, the mapping $S \to Y : x \mapsto f(x)$ and refer to $g_{|S}$ as the restriction of g to S.

For $\mathcal{C}$ a class, we write $\{X_i\}_{i:I\to\mathcal{C}}$ to denote a collection of distinguished members of $\mathcal{C}$ that are indexed by another class I and refer to it as an indexed family of elements of $\mathcal{C}$ if making explicit reference to I is not necessary. Unless differently stated, all indexed families considered in this paper are implicitly indexed by sets, and the above notation is simplified, as is usual, to $\{X_i\}_{i\in I}$ whenever $\mathcal{C}$ is implied by the context, and indeed to $\{X_i\}_{i=1}^n$ if $|I| = n$ for some $n \in \mathbb{N}$ (an empty family if $n = 0$). Finally, for X a set, we write $|X|$ for its cardinality.

1.2. **Organization.** In Section 2, after having recalled some rudiments of model theory and given definitions useful to adapt them to our specific needs, we give the notion of structure used through the paper. Then, in Section 3, we introduce prealgebraic [resp. subalgebraic] theories and prenorms [resp. subnorms] and prove the main result of the paper (i.e. Proposition 3.1), subsequently presenting the category of prealgebraic [resp. subalgebraic] **K**-models relative to a certain family of prealgebraic [resp. subalgebraic] theories, where **K** is a category with finite products. Section 4 discusses prenormed [resp. subnormed] models and Section 5 shows how these are ultimately an abstraction of familiar normed structures, such as normed groups and valuated rings, by a number of examples. Lastly, Appendix A provides a short introductory overview to category theory. The intent is twofold. On the one hand, we feel necessary to fix, once and for all, basic notation and terminology that we are going to use, both here and in future work, to deal with categories. On the other, this article has been motivated by research in the field of numerical analysis, and therefore it aims to attract the interest not only, say, of categorists but also of non-specialists in the area.

2. Finitary single-sorted signatures and structures thereof

In the traditional language of model theory and first-order logic [19], a (finitary single-sorted) signature, or type, is a triple $\sigma = (\Sigma_\mathrm{f}, \Sigma_\mathrm{r}, \mathrm{ar})$, where Σ_f and Σ_r are disjoint sets not including logical symbols of the underlying formal language and ar is a map $\Sigma_\mathrm{f} \cup \Sigma_\mathrm{r} \to \mathbb{N}^+$. The members of Σ_f are called function symbols, those of Σ_r relation symbols. For each symbol $\zeta \in \Sigma_\mathrm{f} \cup \Sigma_\mathrm{r}$, $\mathrm{ar}(\zeta)$ is referred to as the ariety of ζ: one says that ζ is an n-ary function symbol if $\zeta \in \Sigma_\mathrm{f}$ and $\mathrm{ar}(\zeta) = n + 1$ and an n-ary relation symbol if $\zeta \in \Sigma_\mathrm{r}$ and $\mathrm{ar}(\zeta) = n$. Note explicitly that we do not allow for nullary relation symbols. A subsignature of σ is any signature $\sigma_0 = (\Sigma_{\mathrm{f},0}, \Sigma_{\mathrm{r},0}, \mathrm{ar}_0)$ such that $\Sigma_{\mathrm{f},0} \subseteq \Sigma_\mathrm{f}$, $\Sigma_{\mathrm{r},0} \subseteq \Sigma_\mathrm{r}$ and ar_0 is the restriction of ar to $\Sigma_{\mathrm{f},0} \cup \Sigma_{\mathrm{r},0}$. In addition to this, we say that σ is algebraic if $\Sigma_\mathrm{r} = \emptyset$ and *balanced* if there exists a bijection $\phi : \Sigma_\mathrm{f} \to \Sigma_\mathrm{r}$.

Remark 1. A balanced signature $\sigma = (\Sigma_\mathrm{f}, \Sigma_\mathrm{r}, \mathrm{ar})$ can be, and will be, systematically represented as $(\{(\varsigma_r, \varrho_r)\}_{r\in R}; \mathrm{ar})$, where $\{\varsigma_r : r \in R\} = \Sigma_\mathrm{f}$ and $\{\varrho_r : r \in R\} = \Sigma_\mathrm{r}$.

Provided that $\sigma_i = (\Sigma_{\mathrm{f},i}, \Sigma_{\mathrm{r},i}, \mathrm{ar}_i)$ is a signature $(i = 1, 2)$, we define a signature homomorphism from σ_1 to σ_2 to be a map $\alpha : \Sigma_{\mathrm{f},1} \cup \Sigma_{\mathrm{r},1} \to \Sigma_{\mathrm{f},2} \cup \Sigma_{\mathrm{r},2}$ such that $\alpha(\Sigma_{\mathrm{f},1}) \subseteq \Sigma_{\mathrm{f},2}$, $\alpha(\Sigma_{\mathrm{r},1}) \subseteq \Sigma_{\mathrm{r},2}$ and $\mathrm{ar}_2(\alpha(\zeta)) = \mathrm{ar}_1(\zeta)$ for every $\zeta \in \Sigma_{\mathrm{f},1} \cup \Sigma_{\mathrm{r},1}$. If so, we write that $\alpha : \sigma_1 \to \sigma_2$ is a signature homomorphism. In addition to this, for $\sigma_0 = (\Sigma_{\mathrm{f},0}, \Sigma_{\mathrm{r},0}, \mathrm{ar}_0)$ a subsignature of σ_1, we say that a signature homomorphism $\alpha_0 : \sigma_0 \to \sigma_2$ is the restriction of α to σ_0 if, regarded as a function, it is the restriction of α to $\Sigma_{\mathrm{f},0} \cup \Sigma_{\mathrm{r},0}$. Lastly, we refer to the signature homomorphism $j : \sigma_0 \to \sigma_1$ sending each $\zeta \in \Sigma_{\mathrm{f},0} \cup \Sigma_{\mathrm{r},0}$ to itself as the canonical injection $\sigma_0 \to \sigma_1$.

Remark 2. Pick $\sigma = (\Sigma_\mathrm{f}, \Sigma_\mathrm{r}, \mathrm{ar})$ to be a signature. For $k, \ell \in \mathbb{N}$, suppose that Σ_f and Σ_r can be respectively partitioned into k families of function symbols $\{\varsigma_{1,r}\}_{r\in R_1}, \ldots, \{\varsigma_{k,r}\}_{r\in R_k}$ and ℓ families of relation symbols $\{\varrho_{1,s}\}_{s\in S_1}, \ldots, \{\varrho_{\ell,s}\}_{s\in S_\ell}$. Then, σ is possibly denoted by

$$(\{\varsigma_{1,r}\}_{r\in R_1}, \ldots, \{\varsigma_{k,r}\}_{r\in R_k}; \{\varrho_{1,s}\}_{s\in S_1}, \ldots, \{\varrho_{\ell,s}\}_{s\in S_\ell}; \mathrm{ar}). \tag{1}$$

On another hand, assume that σ is balanced and let $\sigma = (\{(\varsigma_r, \varrho_r)\}_{r\in\Sigma_\mathrm{f}}; \mathrm{ar})$ (see Remark 1). Admit that there exists $k \in \mathbb{N}$ such that $\{(\varsigma_r, \varrho_r)\}_{r\in\Sigma_\mathrm{f}}$ can be partitioned into k collections of the form $\{(\varsigma_{1,r}, \varrho_{1,r})\}_{r\in R_1}, \ldots, \{(\varsigma_{k,r}, \varrho_{k,r})\}_{r\in R_k}$. Then, we possibly write σ as

$$(\{(\varsigma_{1,r}, \varrho_{1,r})\}_{r\in R_1}; \ldots; \{(\varsigma_{k,r}, \varrho_{k,r})\}_{r\in R_k}; \mathrm{ar}). \tag{2}$$

These notations are further simplified, in the most obvious way, whenever a family of symbols consists of one element (i.e. is a singleton), to the extent of writing, for instance, $(+, \star, 1; \leq, \sim; \mathrm{ar})$ in place of $\sigma = (\{+, \star, 1\}, \{\leq, \sim\}, \mathrm{ar})$ or $(-, \preceq; \star, \simeq; \mathrm{ar})$ instead of $(\{-, \preceq\}; \{\star, \simeq\}; \mathrm{ar})$.

(One-sorted finitary) signatures form a category, denoted by $\mathbf{Sgn}_1$. This has the class of signatures as objects and all triples of the form (σ, τ, α) as morphisms, with σ and τ being signatures and $\alpha : \sigma \to \tau$ a signature homomorphism. As is customary, when there is no likelihood of ambiguity, we will use α as a shorthand of (σ, τ, α). The composition of two morphisms (σ, τ, α) and (τ, ω, β) is defined by the triple $(\sigma, \omega, \beta \circ_{\mathbf{Set}} \alpha)$. The local identities and the maps of source and target are the obvious ones.

With this in hand, let us assume henceforth that $\mathbf{K}$ is a category with (all) finite products (see Appendix A for notation and terminology). In our understanding, a (finitary single-sorted) structure over $\mathbf{K}$, or $\mathbf{K}$-structure, is then any 4-uple $\mathfrak{A} = (A, \chi, \sigma, \mathfrak{i})$ consisting of

(i) an object A of $\mathbf{K}$, referred to as the carrier of the structure and denoted by $|\mathfrak{A}|$.
(ii) a *sorting function* $\chi : \mathbb{N} \to \bigcup_{k\in\mathbb{N}} \prod_\mathbf{K} \{A\}_{i=1}^k$ such that $\chi(n) \in \prod_\mathbf{K} \{A\}_{i=1}^k$ and $\chi(1) = (A, \mathrm{id}_\mathbf{K}(A))$, i.e., $\chi(n)$ is a pair (P_n, Π_n), with P_n an object in the isomorphism class of $\prod_\mathbf{K} \{A\}_{i=1}^k$ and Π_n a distinguished set of projections $P_n \to A$ (see Remark 19).
(iii) a (finitary single-sorted) signature $\sigma = (\Sigma_\mathrm{f}, \Sigma_\mathrm{r}, \mathrm{ar})$.
(iv) an *interpretation function* $\mathfrak{i} : \Sigma_\mathrm{f} \cup \Sigma_\mathrm{r} \to \hom(\mathbf{K})$ sending an n-ary function symbol to an arrow of $\hom_\mathbf{K}(P_n, A)$ and an n-ary relation symbol to a monomorphism $r : R \to P_n$ such that (r, Π_n) is an n-ary relation on A (see Definition 5), where $(P_n, \Pi_n) = \chi(n)$.

The interpretation $\mathfrak{i}(\varsigma)$ of a nullary function symbol ς of σ will be called a constant of $\mathfrak{A}$ (essentially because it can be identified with a distinguished element of A when $\mathbf{K} = \mathbf{Set}$).

Remark 3. In the sequel, dealing with a $\mathbf{K}$-structure $\mathfrak{A} = (A, \chi, \sigma, \mathfrak{i})$, we abuse notation and forget about χ, to the extent of writing A^n in place of $\chi(n)$, with the implicit arrangement that this is only a shorthand for denoting a distinguished pair $(P_n, \Pi_n) \in \prod_\mathbf{K} \{A\}_{i=1}^k$, and indeed for referring to each one of its two components according as appropriate. In particular, given n

parallel points $a_1, a_2, \ldots, a_n \in A$, we write $(a_1, a_2, \ldots, a_n)$ for $(a_1, a_2, \ldots, a_n)_{A^n}$ (see Remark 21). Thus, to describe a **K**-structure, we use a 3-uple instead of a 4-uple and subsequently omit any further reference to any sorting function. In addition, as far as there is no danger of confusion, we do not make any notational distinction between a symbol ζ of σ and its interpretation under $\mathfrak{i}$, to the extent of using ζ for $\mathfrak{i}(\zeta)$: This will be especially the case when ζ is the typical function symbol of a constant or that of an operation, such as $+$, $\star$, 0, or 1, or the usual relation symbol of a preorder, such as $\leq$ or $\preceq$. In these circumstances, for $a, b \in A$, we will write, e.g., $a + b$ in place of $\mathfrak{i}(+)(a, b)$ and $a \leq b$ for $(a, b) \in \mathfrak{i}(\leq)$ without additional explanation (see Remark 21).

Remark 4. Still for the sake of simplifying our notation, suppose that $\mathfrak{A}_i = (A_i, \sigma_i, \mathfrak{i}_i)$ is a **K**-structure ($i = 1, 2$), let φ be a morphism $A_1 \to A_2$ of **K**, and pick $n \in \mathbb{N}$. Then, we agree to denote by φ^n the arrow $\prod_{\mathbf{K}} \langle \pi_{1,j} \circ_{\mathbf{K}} \varphi, \pi_{2,j} \rangle_{j=1}^n$ (indeed a morphism $A_1^n \to A_2^n$), where $\{\pi_{i,j}\}_{j=1}^n$ is the set of canonical projections associated with A_i^n according to Remark 3.

Remark 5. In the cases considered below to work out the basics of the abstract theory of normed structures, we will restrict ourselves to **K**-structures of type $(A, \sigma, \mathfrak{i})$, where σ is balanced and each relation symbol ϱ of σ, if any is present, is binary and its interpretation is a preorder [resp. a partial order] on A, in such a way that $(A, \mathfrak{i}(\varrho))$ is an object of $\mathbf{Pre}(\mathbf{K})$ [resp. $\mathbf{Pos}(\mathbf{K})$], i.e., a preordered object [resp. a pod] of **C** (see Definition 6). When this occurs, $\mathfrak{A}$ will be referred to as a prealgebraic [resp. subalgebraic] **K**-structure, and indeed as an algebraic **K**-structure if $\mathfrak{i}(\varrho)$ is the equality relation on A for every $\varrho \in \Sigma_{\mathrm{r}}$. Algebraic structures over **Set** are precisely the (finitary single-sorted) structures traditionally studied by universal algebra.

Remark 6. Say that $\mathfrak{A} = (A, \sigma, \mathfrak{i})$ is a **K**-structure, with $\sigma = (\Sigma_{\mathrm{f}}, \Sigma_{\mathrm{r}}, \mathrm{ar})$, and assume that, for some $k, \ell \in \mathbb{N}$, it is possible to partition Σ_{f} and Σ_{r}, each in turn, into k families of function symbols $\{\varsigma_{1,r}\}_{r \in R_1}, \ldots, \{\varsigma_{k,r}\}_{r \in R_k}$ and ℓ families of relation symbols $\{\varrho_{1,s}\}_{s \in S_1}, \ldots, \{\varrho_{\ell,s}\}_{s \in S_\ell}$ (see Remark 2). In this case, $\mathfrak{A}$ is possibly represented by

$$(A; \{\varsigma_{1,r}\}_{r \in R_1}, \ldots, \{\varsigma_{k,r}\}_{r \in R_k}; \{\varrho_{1,s}\}_{s \in S_1}, \ldots, \{\varrho_{\ell,s}\}_{s \in S_\ell}). \tag{3}$$

On another hand, admit that σ is balanced and let $\sigma = (\{(\varsigma_r, \varrho_r)\}_{r \in \Sigma_{\mathrm{f}}}; \mathrm{ar})$ (see Remark 1). Suppose that there exists $k \in \mathbb{N}$ such that $\{(\varsigma_r, \varrho_r)\}_{r \in \Sigma_{\mathrm{f}}}$ can be partitioned into k collections of the form $\{(\varsigma_{1,r}, \varrho_{1,r})\}_{r \in R_1}, \ldots, \{(\varsigma_{k,r}, \varrho_{k,r})\}_{r \in R_m}$. Then, we possibly denote $\mathfrak{A}$ by

$$(A; \{(\varsigma_{1,r}, \varrho_{1,r})\}_{r \in R_1}; \ldots; \{(\varsigma_{k,r}, \varrho_{k,r})\}_{r \in R_k}). \tag{4}$$

These notations are further simplified, in the most obvious way, if a family of symbols is a singleton, to the degree of writing, e.g., $(A; +, \star, 1; \leq, \sim)$ in place of $(A, \sigma, \mathfrak{i})$ provided $\sigma = (\{+, \star, 1\}, \{\leq, \sim\}, \mathrm{ar})$ or $(A; -, \preceq; \star, \simeq)$ instead of $(A, \sigma, \mathfrak{i})$ for $\sigma = (\{-, \star\}, \{\preceq, \simeq\}, \mathrm{ar})$.

3. First-order prealgebraic theories and prenorms

Upon these premises, take **K** to be a category with finite products and $\sigma = (\Sigma_{\mathrm{f}}, \mathrm{r}, \mathrm{ar})$ a (finitary single-sorted) signature and fix an infinite set V of variables from the underlying formal language $\mathcal{L}$. The role of V is not that important here, and we could in fact assume that V includes all the variables of $\mathcal{L}$. Yet, explicit reference to the variables to use in quantification will be a key issue for the development of the theory of many-sorted normed structures in the second part of this work, and this is the main motivation beyond most of our non-standard notational choices.

We denote $\langle V; \sigma \rangle$ the set of all (well-formed) formulas generated by combining, according to the formation rules of first-order logic, the variables of V, the symbols of σ and those of $\mathcal{L}$ which are not logical variables. One says that a **K**-structure $\mathfrak{A} = (A, \sigma, \mathfrak{i})$ satisfies a formula

$\phi \in \langle V;\sigma\rangle$ of n arguments $x_1, x_2, \ldots, x_n \in V$ if $\phi(\mathfrak{A})$, i.e., the interpretation of ϕ over $\mathfrak{A}$, is a true statement (which is compactly written as $\mathfrak{A} \models \phi$). Here, $\phi(\mathfrak{A})$ is obtained from ϕ by

(i) replacing each variable x_i with a point $A^0 \to A$ and each $\zeta \in \Sigma_{\rm f} \cup \Sigma_{\rm r}$ occurring in the expression of ϕ with its interpretation under $\mathfrak{i}$.

(ii) interpreting expressions of the form $\mathfrak{i}(\varsigma)(a_1, a_2, \ldots, a_n)$, where $\varsigma \in \Sigma_{\rm f}$ is an n-ary function symbol and $a_1, a_2, \ldots, a_n$ are points $A^0 \to A$ of $\mathbf{K}$, according to Remarks 3.

(iii) interpreting expressions of the form $\mathfrak{i}(\varrho)(a_1, a_2, \ldots, a_n)$, where $\varrho \in \Sigma_{\rm r}$ is an n-ary relation symbol and $a_1, a_2, \ldots, a_n$ are points $A^0 \to A$ of $\mathbf{K}$, as $(a_1, a_2, \ldots, a_n) \in \mathfrak{i}(\varrho)$, according to Remarks 3 and 21.

This is enough to unambiguously specify the meaning of $\phi(\mathfrak{A})$ for the fact that points and their parallelism are preserved under morphisms (see again Remark 21). A σ-theory, or a theory of type σ, in the variables V is then any triple $T = (V, \sigma, \Xi)$ such that Ξ is a (possibly empty) subset of $\langle V;\sigma\rangle$, while a $\mathbf{K}$-model $\mathfrak{A}$ of T, or equivalently a model of T over $\mathbf{K}$, is a $\mathbf{K}$-structure $(A, \sigma, \mathfrak{i})$ that satisfies every axiom $\phi \in \Xi$. Such a condition is equivalently stated by writing $\mathfrak{A} \models T$ and saying that $\mathfrak{A}$ satisfies T: in this respect, σ will be also referred to as the signature of $(T, \mathfrak{A})$. If $T = (V, \sigma, \Xi)$ is a theory, a subtheory of T is any theory $T_s = (V, \sigma_s, \Xi_s)$ such that σ_s is a subsignature of σ and $\Xi_s = \Xi \cap \langle V;\sigma_s\rangle$, while a $\mathbf{K}$-submodel of T is a $\mathbf{K}$-model of a subtheory of T. If T_s is a subtheory of T, we write $T_s \le T$ and possibly say that T is an extension, or a supertheory, of T_s.

Remark 7. Let σ include, among its functional symbols, two binary symbols $\vee$ and $\wedge$, a unary symbol u and a nullary symbol e. Then, consider the following formulas from $\langle V;\sigma\rangle$:

(A.1) $\forall x, y, z \in V : (x \vee y) \vee z = x \vee (y \vee z)$.
(A.2) $\forall x \in V : x \vee e = e \vee x = x$.
(A.3) $\forall x \in V : x \vee u(x) = u(x) \vee x = e$.
(A.4) $\forall x, y, z \in V : x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.
(A.5) $\forall x, y, z \in V : (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.

We refer, as is usual, to (A.1) as the axiom of associativity for the symbol $\vee$; to (A.2) as the axiom of neutrality for the pair $(\vee, e)$; to (A.3) as the axiom of inverses for the triple $(\vee, u, e)$; to (A.4) and (A.5), respectively, as the axioms of left and right distributiveness of $\wedge$ over $\vee$. We list them here for future reference, they will be used later to deal with examples in Section 5.

Definition 1. We say that a theory $T = (V, \sigma, \Xi)$ is *prealgebraic* if σ is a balanced signature of type $(\Sigma_{\rm f}, \Sigma_{\rm r}, {\rm ar})$, where all relation symbols are binary and, for each $\varrho \in \Sigma_{\rm r}$, the axioms of T include at least the axiom of reflexivity: $\forall x \in V : \varrho(x, x)$, and the axiom of transitivity:

$$\forall x, y, z \in V : (\varrho(x, y) \text{ AND } \varrho(y, z)) \implies \varrho(x, z).$$

In addition to this, T will be called *subalgebraic* if it is prealgebraic and, for every $\varrho \in \Sigma_{\rm r}$, Ξ contains also the axiom of antisymmetry:

$$\forall x, y \in V : (\varrho(x, y) \text{ AND } \varrho(y, x)) \implies (x = y).$$

Lastly, T is called algebraic if $\Sigma_{\rm r}$ is empty (no relational symbols are permitted) and algebraic over $\mathbf{K}$ if, for each $\mathbf{K}$-model $\mathfrak{A} = (A, \sigma, \mathfrak{i})$ of T, there is one more $\mathbf{K}$-model $\mathfrak{A}_{\rm a} = (A, \sigma, \mathfrak{i}_{\rm a})$ of T such that $\mathfrak{i}_{\rm a}(\varsigma) = \mathfrak{i}(\varsigma)$ for each $\varsigma \in \Sigma_{\rm f}$ and $\mathfrak{i}_{\rm a}(\varrho)$ is the equality relation on A for all $\varrho \in \Sigma_{\rm r}$. Then, $\mathfrak{A}_{\rm a}$ is an algebraic $\mathbf{K}$-structure (see Remark 5), referred to as an algebraization of $\mathfrak{A}$, on the one hand, and an algebraic $\mathbf{K}$-model of T, on the other.

Note that, if T is prealgebraic [resp. subalgebraic], any relation symbol of its signature will be interpreted, in any possible $\mathbf{K}$-model $\mathfrak{A}$ of T, as a preorder [resp. a partial order] on $|\mathfrak{A}|$.

Definition 2. A $\mathbf{K}$-model $\mathfrak{A} = (A, \sigma, \mathfrak{i})$ of a theory $T = (V, \sigma, \Xi)$, with $\sigma = (\Sigma_\mathrm{f}, \Sigma_\mathrm{r}, \mathrm{ar})$, is said *pivotal* if the symbols of Σ_r have all the same ariety and either $\Sigma_\mathrm{r} = \emptyset$ or there exists $\varrho_0 \in \Sigma_\mathrm{r}$ such that $\mathfrak{i}(\varrho) \subseteq \mathfrak{i}(\varrho_0)$ for every $\varrho \in \Sigma_\mathrm{r}$. When this happens, $\mathfrak{i}(\varrho_0)$ is called the *pivot* of $\mathfrak{A}$.

Pivotal models will be crucial, later in this section, for the definition of the categories of prenormed and subnormed structures over a fixed "target" (see Section 4).

Remark 8. A subalgebraic theory is always a prealgebraic theory. More interestingly, there is a canonical way to identify a $\mathbf{K}$-model $\mathfrak{A} = (A, \sigma, \mathfrak{i})$ of an algebraic theory $T = (V, \sigma, \Xi)$ with a $\mathbf{K}$-model of a subalgebraic theory (in the same variables), so that any algebraic theory can be definitely identified with an subalgebraic theory. To see how, let $\sigma = (\Sigma_\mathrm{f}, \emptyset, \mathrm{ar})$. For each $\varsigma \in \Sigma_\mathrm{f}$, consider a binary relation symbol $\varrho_\varsigma \notin \Sigma_\mathrm{f}$, not already comprised among the basic symbols of the underlying logic, and set $\Sigma_\mathrm{r} := \{\varrho_\varsigma\}_{\varsigma \in \Sigma_\mathrm{f}}$. Extend ar to the function $\mathrm{ar}_e : \Sigma_\mathrm{f} \cup \Sigma_\mathrm{r} \to \mathbb{N}$ by taking $\mathrm{ar}_e(\varrho) := 2$ for every $\varrho \in \Sigma_\mathrm{r}$ and define $\sigma_e := (\Sigma_\mathrm{f}, \Sigma_\mathrm{r}, \mathrm{ar}_e)$. Lastly, expand Ξ to a larger set of axioms, namely Ξ_e, in such a way as to include all and only the axioms of reflexivity, symmetry and transitivity relative to every relation symbol $\varrho \in \Sigma_\mathrm{r}$. Then, $T_e = (V, \sigma_e, \Xi_e)$ is a subalgebraic (and hence prealgebraic) theory and $\mathfrak{A}$ can be identified with the model $\mathfrak{A}_e = (A, \sigma_e, \mathfrak{i}_e)$ of T_e defined by assuming that $\mathfrak{i}_e = \mathfrak{i}$ on Σ_f and $\mathfrak{i}_e(\varrho)$ is the equality relation on A for each $\varrho \in \Sigma_\mathrm{r}$. On another hand, say that T is an *arbitrary* theory. Then, a slight modification of the above arguments shows that it is always possible to find a smallest prealgebraic [resp. subalgebraic] theory in the same variables and with the same signature as T, where "smallest" must be intended as "with the fewest possible axioms". This will be denoted by $\sharp_p T$ [resp. $\sharp_s T$] and called the prealgebraic [resp. subalgebraic] embodiment of T.

Remark 9. When the set of variables V is well understood from the context, we will use simply (σ, Ξ) in place of (V, σ, Ξ) to indicate a theory in the variables V.

If $P \in \mathrm{obj}(\mathbf{K})$ and $\mathcal{Q} = (Q, \le_Q)$ is a preordered object of $\mathbf{K}$ (see Definition 6), we convey to denote by $\wp(P, \mathcal{Q})$ the preorder induced on $\hom_\mathbf{K}(P, Q)$ by $\le_Q$ as follows: If $f, g \in \hom_\mathbf{K}(P, Q)$, we let $(f, g) \in \wp(P, \mathcal{Q})$ if and only if $f(x) \le_Q g(x)$ for all $x \in P$. Observe that, in the special case where $\mathbf{K} = \mathbf{Set}$, $\wp(P, \mathcal{Q})$ represents, up to an isomorphism, the preorder corresponding to the exponential [1, Chapter 6] of the pair $(\mathcal{P}, \mathcal{Q})$ in $\mathbf{Pre}(\mathbf{Set})$ for $\mathcal{P}$ being the trivial poset on P (see Example C.3 in Appendix A). Also, note that $\wp(P, \mathcal{Q})$ is, in fact, a partial order on $\hom_\mathbf{K}(P, Q)$ when $\mathcal{Q}$ is a pod of $\mathbf{K}$ and $\mathbf{K}$ is such that two arrows $f, g : A \to B$ are equal if $f(a) = g(a)$ for all $a \in A$: this is, e.g., the case for $\mathbf{Set}$, and more generally whenever the terminal objects of $\mathbf{K}$ are generators [3, Section 4.5].

Lemma 3.1. *Given that $P \in \mathrm{obj}(\mathbf{K})$ and $\mathcal{Q} = (Q, \le_Q)$ and $\mathcal{R} = (R, \le_R)$ are preordered objects of $\mathbf{K}$, let $f, f_1, f_2 \in \hom_\mathbf{K}(P, Q)$ and $g, g_1, g_2 \in \hom_\mathbf{K}(Q, R)$. The following hold:*

(i) *If $(g_1, g_2) \in \wp(Q, \mathcal{R})$, then $(g_1 \circ_\mathbf{K} f, g_2 \circ_\mathbf{K} f) \in \wp(P, \mathcal{R})$.*

(ii) *If $(f_1, f_2) \in \wp(P, \mathcal{Q})$ and $g \in \hom_{\mathbf{Pre}(\mathbf{K})}(\mathcal{Q}, \mathcal{R})$, then $(g \circ_\mathbf{K} f_1, g \circ_\mathbf{K} f_2) \in \wp(P, \mathcal{R})$.*

Proof. The first claim is obvious. As for the second, pick $x \in P$. Since $(f_1, f_2) \in \wp(P, \mathcal{Q})$, it is $f_1(x) \le_Q f_2(x)$, from which $g(f_1(x)) \le_R g(f_2(x))$, for g is a monotonic function $\mathcal{Q} \to \mathcal{R}$. By the arbitrariness of $x \in P$, this completes the proof. ∎

Now, in what follows, let $T_i = (V, \sigma_i, \Xi_i)$ be a prealgebraic [resp. subalgebraic] theory and $\mathfrak{A}_i = (A_i, \sigma_i, \mathfrak{i}_i)$ a $\mathbf{K}$-model of T_i $(i = 1, 2)$, with $\sigma_i = (\{\varsigma_{r,i}, \varrho_{r,i}\}_{r \in R_i}; \mathrm{ar}_i)$. It is then possible to regard a signature homomorphism $\alpha : \sigma_1 \to \sigma_2$ as a pair (α_1, α_2) of maps $\alpha_1, \alpha_2 : R_1 \to R_2$ by imposing that $\varsigma_{\alpha_1(r),2} = \alpha(\varsigma_{r,1})$ and $\varrho_{\alpha_2(r),2} = \alpha(\varrho_{r,1})$ for each $r \in R_1$. Hence, we systematically

abuse notation and identify α with its "components" α_1 and α_2, to the extent of writing $\varsigma_{\alpha(r),2}$ in place of $\varsigma_{\alpha_1(r),2}$ and $\varrho_{\alpha(s),2}$ for $\varrho_{\alpha_2(s),2}$. In addition to this, set $\mathcal{A}_{r,i}$ equal to $(A_i, \varrho_{r,i})$ for every $r \in R_i$ (while having in mind Remark 3). The following definition is fundamental:

Definition 3. We say that $\mathfrak{A}_1$ is prehomomorphic [resp. subhomomorphic] to $\mathfrak{A}_2$ if there exist $\alpha \in \hom_{\mathbf{Sgn}_1}(\sigma_1, \sigma_2)$ and $\varphi \in \hom_{\mathbf{K}}(A_1, A_2)$ such that, for each $r \in R_1$,

(i) $(\varphi \circ_{\mathbf{K}} \varsigma_{r,1}, \varsigma_{\alpha(r),2} \circ_{\mathbf{K}} \varphi^n) \in \wp(A_1^n, \mathcal{A}_{\alpha(r),2})$ (see Remark 4);

(ii) φ is a monotonic arrow $\mathcal{A}_{r,1} \to \mathcal{A}_{\alpha(r),2}$ in $\mathbf{Pre}(\mathbf{K})$ (see Example C.3 in Appendix A),

where $n := \mathrm{ar}_1(\varsigma_{r,1})$ (see Remark). Then, we call $\Phi := (\alpha, \varphi)$ a **K**-prenorm [resp. **K**-subnorm] or write that $\Phi : \mathfrak{A}_1 \to \mathfrak{A}_2$ is a prenorm [resp. subnorm] of **K**-models. In particular, a **K**-prenorm $\mathfrak{A}_1 \to \mathfrak{A}_2$ is said a **K**-homomorphism if $\mathfrak{i}_2(\varrho_{\alpha(r),2})$ is the equality on A_2 for each $r \in R_1$.

Remark 10. Definition 3 returns exactly the standard notion of a homomorphism of algebraic structures, as given in the framework of universal algebra, in the special case where T_i is an algebraic theory over **Set** and $T_1 = T_2$.

Remark 11. Let $T_\emptyset$ denote the "empty theory" $(V, \sigma_\emptyset, \emptyset)$, where $\sigma_\emptyset = (\emptyset, \emptyset, \emptyset)$. Clearly, $T_\emptyset$ is a subalgebraic (indeed algebraic) theory and its **K**-models are all and only the triples $(A, \sigma_\emptyset, \emptyset)$ such that $A \in \mathrm{obj}(\mathbf{K})$. Therefore, for T another prealgebraic [resp. subalgebraic] theory in the variables V, $\mathfrak{A}_\emptyset$ a **K**-model of $T_\emptyset$ and $\mathfrak{A}$ a **K**-model of T, the **K**-prenorms [resp. **K**-subnorms] $\mathfrak{A}_\emptyset \to \mathfrak{A}$ are all and only the pairs $(\emptyset, \varphi)$ for which φ is a morphism $|\mathfrak{A}_\emptyset| \to |\mathfrak{A}|$ in **K**.

Remark 12. Except for those required to turn T_1 and T_2 into prealgebraic [resp. subalgebraic] theories, the definition of a **K**-prenorm [resp. **K**-subnorm] $\mathfrak{A}_1 \to \mathfrak{A}_2$ does not depend at all on the axioms that $\mathfrak{A}_1$ and $\mathfrak{A}_2$ have to satisfy as models of T_1 and T_2, respectively. This ultimately means that other axioms, if any is present, do not play an active role in the foundations of the abstract theory so far developed. Rather, they (can) contribute to determining "extrinsic" properties of prenorms (and, later, prenormed structures), i.e., properties complementary to the inherent ones stemming directly from their very definition.

Remark 13. Clearly, for a **K**-prenorm $\mathfrak{A}_1 \to \mathfrak{A}_2$ to exist, it is necessary that any n-ary function symbol of σ_1 has a corresponding n-ary function symbol in σ_2, though it is not necessary that σ_1 is smaller than σ_2 (in the sense that the former contains less symbols than the latter).

Remark 14. If $R_1 = \emptyset$, the pair (α, φ) is a **K**-prenorm $\mathfrak{A}_1 \to \mathfrak{A}_2$ if and only if $\alpha = (\sigma_1, \sigma_2, \emptyset)$ and $\varphi \in \hom_{\mathbf{K}}(A_1, A_2)$ is monotonic in the sense of condition (ii) of Definition 3. Lastly, for $\mathfrak{A}_1$ an algebraic **K**-model of T_1, (α, φ) is a **K**-prenorm $\mathfrak{A}_1 \to \mathfrak{A}_2$ if and only if it satisfies condition (i) in the aforementioned definition (since the other, in this case, is automatically fulfilled).

Remark 15. Suppose that (α, φ) is a **K**-prenorm $\mathfrak{A}_1 \to \mathfrak{A}_2$ and pick an index $r \in R_1$, if any exists, such that $\varsigma_{r,1}$ is (interpreted as) a nullary operation of $\mathfrak{A}_1$. It then follows from Definition 3 that $(\varphi(c_{r,1}), c_{\alpha(r),2}) \in \varrho_{\alpha(r),2}$, where $c_{r,1} := \varsigma_{r,1} \circ_{\mathbf{K}} \mathrm{id}_{\mathbf{K}}(A_1^0)$ is a distinguished point of A_1 and $c_{r,2} := \varsigma_{\alpha(r),2} \circ_{\mathbf{K}} \mathrm{id}_{\mathbf{K}}(A_2^0)$ a distinguished point of A_2. In particular, if $\varrho_{\alpha(r),2}$ is antisymmetric, thus a partial order on A_2, and $c_{\alpha(r),2}$ is the least element of $(A_2, \varrho_{\alpha(r),2})$, in the sense that $(c_{\alpha(r),2}, a) \in \varrho_{\alpha(r),2}$ for every $a \in A_2$, this implies $\varphi(c_{r,1}) = c_{\alpha(r),2}$. On account of the worked examples examined in Section 5, such a result represents a minor but attractive byproduct of the framework set up in this work. In the ultimate analysis, it shows that there is no need to *assume*, say, that a group norm or a ring valuation, as defined in the traditional setting by taking them to be valued in $\mathbb{R}_0^+$ (cf. Examples E.3 and E.5 in Section 5), preserve the additive identities, for this is nothing but a *consequence* of the inherent properties of subnorms.

Definition 4. Assume that $\mathfrak{A}_2$ is pivotal and denote its pivot by $\leq$. Take ς_2 to be a nullary function symbol in σ_2 (if any exists) and $\Phi = (\alpha, \varphi)$ a $\mathbf{K}$-prenorm $\mathfrak{A}_1 \to \mathfrak{A}_2$. We say that Φ is

(i) upward [resp. downward] semidefinite with respect to ς_2 if $(\varsigma_2 \circ_{\mathbf{K}} \varphi^0)(a) \leq \varphi(a)$ [resp. $\varphi(a) \leq (\varsigma_2 \circ_{\mathbf{K}} \varphi^0)(a)$] for every $a \in A_1$;

(ii) upward [resp. downward] definite (with respect to ς_2) if it is upward [resp. downward] semidefinite and $(\varsigma_2 \circ_{\mathbf{K}} \varphi^0)(a) \neq \varphi(a)$ for all $a \in A_1 \setminus \bigcup_{\varsigma_1 \in \alpha^{-1}(\varsigma_2)} \varsigma_1$, which is equivalently expressed by writing that $(\varsigma_2 \circ_{\mathbf{K}} \varphi^0)(a) \lneq \varphi(a)$ [resp. $\varphi(a) \lneq (\varsigma_2 \circ_{\mathbf{K}} \varphi^0)(a)$] for every $a \in A_1 \setminus \bigcup_{\varsigma_1 \in \alpha^{-1}(\varsigma_2)} \varsigma_1$.

(iii) indefinite (with respect to ς_2) if it is neither upward nor downward semidefinite.

On another hand, we say that Φ is trivial if $\varphi(a) = \varphi(b)$ for all parallel points $a, b \in A_1$.

Upward (semi)definiteness abstracts and generalizes one of the most basic properties of standard norms, to wit, positive (semi)definiteness. More than this, Definition 4 suggests that, at least in principle, (semi)definitiness of norms has nothing really special to do with the additive identities in group-like, ring-like or module-like structures, as one might naively conclude from the classical perspective. Rather, it is an issue related to constants, all constants: Which one of them is more significant than the others strongly depends on the case at hand.

Proposition 3.1. *Let $T_i = (V, \sigma_i, \Xi_i)$ be a prealgebraic [resp. subalgebraic] theory and $\mathfrak{A}_i = (A_i, \sigma_i, \mathfrak{i}_i)$ a $\mathbf{K}$-model of T_i ($i = 1,2,3$). Suppose $\Phi = (\alpha, \varphi)$ is a $\mathbf{K}$-prenorm [resp. $\mathbf{K}$-subnorm] $\mathfrak{A}_1 \to \mathfrak{A}_2$ and $\Psi = (\beta, \psi)$ a $\mathbf{K}$-prenorm [resp. $\mathbf{K}$-subnorm] $\mathfrak{A}_2 \to \mathfrak{A}_3$. Finally, take $\gamma := \beta \circ_{\mathbf{Sgn}_1} \alpha$ and $\vartheta := \psi \circ_{\mathbf{K}} \varphi$. Then, $\Theta = (\gamma, \vartheta)$ is a $\mathbf{K}$-prenorm [resp. -subnorm] $\mathfrak{A}_1 \to \mathfrak{A}_3$.*

Proof. Assume $\sigma_i = (\{(\varsigma_{r,i}, \varrho_{r,i})\}_{r \in R_i}; \mathrm{ar}_i)$ and, for every $r \in R_i$, take $\mathcal{A}_{r,i}$ to be the preordered [resp. partially ordered] object $(A_i, \varrho_{r,i})$ of $\mathbf{K}$. Since $\mathbf{Sgn}_1$ and $\mathbf{Pre}(\mathbf{K})$ [resp. $\mathbf{Pos}(\mathbf{K})$] are categories, γ is obviously a signature homomorphism $\sigma_1 \to \sigma_2$ and ϑ a monotonic arrow $\mathcal{A}_{r,1} \to \mathcal{A}_{\gamma(r),3}$ for every $r \in R_1$. Thus, it is left to prove that $(\vartheta \circ_{\mathbf{K}} \varsigma_{r,1}, \varsigma_{\gamma(r),3} \circ_{\mathbf{K}} \vartheta^n) \in \wp(A_1^n, \mathcal{A}_{\gamma(r),3})$ for each n-ary function symbol $\varsigma_{r,1} \in \sigma_1$. For this purpose, pick $r \in R_1$ and set $n := \mathrm{ar}_1(\varsigma_{r,1})$. Since, by hypothesis, $(\varphi \circ_{\mathbf{K}} \varsigma_{r,1}, \varsigma_{\alpha(r),2} \circ_{\mathbf{K}} \varphi^n) \in \wp(A_1^n, \mathcal{A}_{\alpha(r),2})$ and ψ is a monotonic morphism $\mathcal{A}_{\alpha(r),2} \to \mathcal{A}_{\gamma(r),3}$, it follows from the second point of Lemma 3.1 that

$$(\psi \circ_{\mathbf{K}} (\varphi \circ_{\mathbf{K}} \varsigma_{r,1}), \psi \circ_{\mathbf{K}} (\varsigma_{\alpha(r),2} \circ_{\mathbf{K}} \varphi^n)) \in \wp(A_1^n, \mathcal{A}_{\gamma(r),3}). \tag{5}$$

By the associativity of $\circ_{\mathbf{K}}$, this in turn is equivalent to

$$(\vartheta \circ_{\mathbf{K}} \varsigma_{r,1}, (\psi \circ_{\mathbf{K}} \varsigma_{\alpha(r),2}) \circ_{\mathbf{K}} \varphi^n) \in \wp(A_1^n, \mathcal{A}_{\gamma(r),3}). \tag{6}$$

On the other hand, again by hypothesis, $(\psi \circ_{\mathbf{K}} \varsigma_{\alpha(r),2}, \varsigma_{\gamma(r),3} \circ_{\mathbf{K}} \psi^n) \in \wp(A_2^n, \mathcal{A}_{\gamma(r),3})$. Hence, the first point of Lemma 3.1 implies that

$$((\psi \circ_{\mathbf{K}} \varsigma_{\alpha(r),2}) \circ_{\mathbf{K}} \varphi^n, (\varsigma_{\gamma(r),3} \circ_{\mathbf{K}} \psi^n) \circ_{\mathbf{K}} \varphi^n) \in \wp(A_1^n, \mathcal{A}_{\gamma(r),3}). \tag{7}$$

Using once more the associativity of $\circ_{\mathbf{K}}$, along with the fact that $\psi^n \circ_{\mathbf{K}} \varphi^n = (\psi \circ_{\mathbf{K}} \varphi)^n = \vartheta^n$, this equation can be rearranged in the form:

$$((\psi \circ_{\mathbf{K}} \varsigma_{\alpha(r),2}) \circ_{\mathbf{K}} \varphi^n, \varsigma_{\gamma(r),3} \circ_{\mathbf{K}} \vartheta^n) \in \wp(A_1^n, \mathcal{A}_{\gamma(r),3}). \tag{8}$$

In the light of Equation (6), it then follows that $(\vartheta \circ_{\mathbf{K}} \varsigma_{r,1}, \varsigma_{\gamma(r),3} \circ_{\mathbf{K}} \vartheta^n) \in \wp(A_1^n, \mathcal{A}_{\gamma(r),3})$, since $\wp(A_1^n, \mathcal{A}_{\gamma(r),3})$ is a preorder on $\hom_{\mathbf{K}}(A_1^n, A_3)$. And this ultimately proves, by the arbitrariness of $r \in R_1$, that (γ, ϑ) is a $\mathbf{K}$-prenorm [resp. $\mathbf{K}$-subnorm] $\mathfrak{A}_1 \to \mathfrak{A}_3$. ■

With this in hand, suppose $\mathfrak{T}$ is a given collection of prealgebraic [resp. subalgebraic] theories in the variables V. We take $\mathcal{C}_\mathrm{o}$ to be the class of all pairs $(T, \mathfrak{A})$ for which $T \in \mathfrak{T}$ and $\mathfrak{A} \models T$, and $\mathcal{C}_\mathrm{h}$ that of all triples $(\mathcal{M}_1, \mathcal{M}_2, \Phi)$ such that $\mathcal{M}_i = (T_i, \mathfrak{A}_i)$, with $T_i \in \mathfrak{T}$ and $\mathcal{F}(\mathfrak{A}_i) \models T_i$, and Φ is a prenorm [resp. subnorm] of $\mathbf{K}$-models $\mathfrak{A}_1 \to \mathfrak{A}_2$. We let s and t be, each in turn, the maps $\mathcal{C}_\mathrm{h} \to \mathcal{C}_\mathrm{o} : (\mathcal{M}_1, \mathcal{M}_2, \Phi) \mapsto \mathcal{M}_1$ and $\mathcal{C}_\mathrm{h} \to \mathcal{C}_\mathrm{o} : (\mathcal{M}_1, \mathcal{M}_2, \Phi) \mapsto \mathcal{M}_2$, while denoting by i the mapping $\mathcal{C}_\mathrm{o} \to \mathcal{C}_\mathrm{h}$ that sends a pair $\mathcal{M} = (T, \mathfrak{A})$ of $\mathcal{C}_\mathrm{o}$, with $T = (\sigma, \Xi)$ and $\mathfrak{A} = (A, \sigma, \mathfrak{i})$, to the triple $(\mathcal{M}, \mathcal{M}, \varepsilon)$ of $\mathcal{C}_\mathrm{h}$ with $\varepsilon := (\mathrm{id}_{\mathbf{Sgn}_1}(\sigma), \mathrm{id}_\mathbf{K}(A))$. Lastly, we specify a partial function $c : \mathcal{C}_\mathrm{h} \times \mathcal{C}_\mathrm{h} \nrightarrow \mathcal{C}_\mathrm{h}$ as follows: Pick $\mathfrak{m} = (\mathcal{M}_1, \mathcal{M}_2, \Phi)$ and $\mathfrak{n} = (\mathcal{N}_1, \mathcal{N}_2, \Psi)$ in $\mathcal{C}_\mathrm{h}$. If $\mathcal{M}_2 \neq \mathcal{N}_1$, then $c(\mathfrak{m}, \mathfrak{n})$ is not defined. Otherwise, in the light of Proposition 3.1, assume $\Phi = (\alpha, \varphi)$ and $\Psi = (\beta, \psi)$ and set $c(\mathfrak{m}, \mathfrak{n}) := (\mathcal{M}_1, \mathcal{N}_2, \Theta)$, where $\Theta := (\beta \circ_{\mathbf{Sgn}_1} \alpha, \psi \circ_\mathbf{K} \varphi)$.

It is then routine to check that $(\mathcal{C}_\mathrm{o}, \mathcal{C}_\mathrm{h}, s, t, i, c)$ is a category. We call it the category of prealgebraic [resp. subalgebraic] $\mathbf{K}$-models of $\mathfrak{T}$. It will be denoted, in general, by $\mathbf{Pnr}_\mathbf{K}(\mathfrak{T})$ [resp. $\mathbf{Snr}_\mathbf{K}(\mathfrak{T})$], and especially written as $\mathbf{Pnr}_\mathbf{K}(T)$ [resp. $\mathbf{Snr}_\mathbf{K}(T)$] in the case where $\mathfrak{T}$ consists of a unique prealgebraic [resp. subalgebraic] theory T (in the variables V). In the latter occurrence, whenever T is implied by the context, we use $\mathfrak{A}$ in place of $(T, \mathfrak{A})$ to mean an object of $\mathbf{Pnr}_\mathbf{K}(T)$ [resp. $\mathbf{Snr}_\mathbf{K}(T)$]. A thorough investigation of the properties of these categories is behind the scope of the present paper: it will, in fact, be the subject of a subsequent article. For the moment, we restrict ourselves to a few trivial remarks and observations.

First, it is clear that $\mathbf{Snr}_\mathbf{K}(\mathfrak{T})$ is contained in $\mathbf{Pnr}_\mathbf{K}(\mathfrak{T})$ as a full subcategory, so that we can partially reduce the study of the former to the study of the latter. Second, suppose $T = (V, \sigma, \Xi)$ is a prealgebraic [resp. subalgebraic] theory and $T_s = (V, \sigma_s, \Xi_s)$ a prealgebraic [resp. subalgebraic] subtheory of T. Then, there exists an obvious "forgetful" functor $\mathcal{C}_{T_s} : \mathbf{Pnr}_\mathbf{K}(T) \to \mathbf{Pnr}_\mathbf{K}(T_s)$ [resp. $\mathcal{C}_{T_s} : \mathbf{Snr}_\mathbf{K}(T) \to \mathbf{Snr}_\mathbf{K}(T_s)$] defined by mapping

(i) a prealgebraic [resp. subalgebraic] $\mathbf{K}$-model $\mathcal{M} = (T, \mathfrak{A})$ of T, with $\mathfrak{A} = (A, \sigma, \mathfrak{i})$, to the pair $(T_s, \mathfrak{A}_s)$, where $\mathfrak{A}_s := (A, \sigma_s, \mathfrak{i}_{|\sigma_s})$;

(ii) a morphism $(\alpha, \varphi) : \mathcal{M}_1 \to \mathcal{M}_2$ of $\mathbf{Pnr}_\mathbf{K}(T)$ [resp. $\mathbf{Snr}_\mathbf{K}(T)$] to the $\mathbf{K}$-prenorm [resp. $\mathbf{K}$-subnorm] $(\alpha_s, \varphi) : \mathcal{C}_{T_s}(\mathcal{M}_1) \to \mathcal{C}_{T_s}(\mathcal{M}_2)$, where α_s is the restriction of α to σ_s.

In particular, $\mathcal{C}_{T_s}$ returns a "forgetful" functor to $\mathbf{K}$ in the extreme case where T_s is the empty theory in the variables V. One question is, then, to establish under which conditions $\mathcal{C}_{T_s}$ admits a left or right adjoint. Nevertheless, this and other properties of $\mathbf{Pnr}_\mathbf{K}(T)$ and $\mathbf{Snr}_\mathbf{K}(T)$ appear to to be strongly dependent on the actual characteristics of T, T_s and $\mathbf{K}$ and will be considered non here but in a separate paper.

4. Prenormed models over a fixed target

Continuing with the notation of the previous section (unless explicitly overridden), assume henceforth that $\mathcal{M}_0 = (T_0, \mathfrak{A}_0)$ is a fixed "target" in $\mathbf{Pnr}_\mathbf{K}(\mathfrak{T})$ and suppose that $\mathfrak{A}_0$ is pivotal: Let us denote its pivot by $\leq$ and set $\mathcal{A}_0 := (|\mathfrak{A}_0|, \leq)$. A prenormed $\mathbf{K}$-model of $\mathfrak{T}$ over $\mathcal{M}_0$ is, then, any pair $\mathsf{M} = (\mathcal{M}, \Phi)$ such that $\mathcal{M}$ is another object of $\mathbf{Pnr}_\mathbf{K}(\mathfrak{T})$ and Φ a $\mathbf{K}$-prenorm $\mathcal{M} \to \mathcal{M}_0$: We refer to Φ as an $\mathcal{M}_0$-valued $\mathbf{K}$-prenorm on $\mathcal{M}$ (or a $\mathbf{K}$-prenorm on $\mathcal{M}$ with values in $\mathcal{M}_0$), and indeed as an $\mathcal{M}_0$-valued $\mathbf{K}$-subnorm on $\mathcal{M}$ (or a $\mathbf{K}$-subnorm on $\mathcal{M}$ with values in $\mathcal{M}_0$) if $\mathcal{M}$ and $\mathcal{M}_0$ are both prenormed $\mathbf{K}$-models of $\mathfrak{T}$. Observe that M can be well identified with the morphism $\Phi : \mathcal{M} \to \mathcal{M}_0$ of $\mathbf{Pnr}_\mathbf{K}(\mathfrak{T})$.

Given a prenormed $\mathbf{K}$-model $\mathsf{M}_i = (\mathcal{M}_i, \Phi_i)$ of $\mathfrak{T}$ over $\mathcal{M}_0$ $(i = 1, 2)$, with $\mathcal{M}_i = (T_i, \mathfrak{A}_i)$ and $\Phi_i = (\alpha_i, \varphi_i)$, we define a $\mathbf{K}$-short morphism $\mathsf{M}_1 \to \mathsf{M}_2$ to be any morphism $(\beta, \psi) : \mathcal{M}_1 \to \mathcal{M}_2$ in $\mathbf{Pnr}_\mathbf{K}(\mathfrak{T})$ such that $\beta \circ_{\mathbf{Sgn}_1} \alpha_2 = \alpha_1$ and $(\varphi_2 \circ_\mathbf{K} \psi, \varphi_1) \in \wp(|\mathfrak{A}_1|, \mathcal{A}_0)$. The terminology is

prompted by the fact that the latter condition is ultimately equivalent to saying, in a much more familiar notation, that $\|\psi(a)\|_2 \le \|a\|_1$ for all $a \in |\mathfrak{A}_1|$, with $\|\cdot\|_i := \varphi_i$. In particular, we write that Ψ is a $\mathbf{K}$-isometry $\mathsf{M}_1 \to \mathsf{M}_2$ if $\|\psi(a)\|_2 = \|a\|_1$ for all $a \in |\mathfrak{A}_1|$.

Remark 16. As shown shortly, prenormed $\mathbf{K}$-models of $\mathfrak{T}$ over $\mathcal{M}_0$ and $\mathbf{K}$-short morphisms thereof give rise to a further category, besides $\mathbf{Pnr_K}(\mathfrak{T})$. In the many-sorted case (discussed in the second part of the present work), this provides a full abstraction of the usual category of left modules over a fixed valuated ring (normed spaces over a fixed valuated field can be viewed as a special case of these), with morphisms given by weakly contractive linear transformations between the underlying vector spaces. This sounds intriguing, for it seems to suggest that, from the perspective of the framework set up in this paper, the "right choice" about the kind of morphisms to be considered in relation to normed structures, based only on abstract nonsense reasoning (and especially regardless of any further considerations relevant to applications), should "naturally" fall on short maps. Simply for the fact that common alternatives available in the "localized" context of normed spaces, such as bounded transformations or continuous functions between the standard topologies induced by the norms on the underlying sets, are ruled out, for they are not even possible, in the setting where we are planting the foundations of the general theory of normed structures.

Lemma 4.1. *Let $\mathsf{M}_i = (\mathcal{M}_i, \Phi_i)$ be a prenormed $\mathbf{K}$-model of $\mathfrak{T}$ over $\mathcal{M}_0$ ($i = 1, 2, 3$) and suppose that Ψ and Θ are, respectively, $\mathbf{K}$-short maps $\mathsf{M}_1 \to \mathsf{M}_2$ and $\mathsf{M}_2 \to \mathsf{M}_3$. The composition of Θ with Ψ in $\mathbf{Pnr_K}(\mathfrak{T})$ is then a $\mathbf{K}$-short map $\mathsf{M}_1 \to \mathsf{M}_3$.*

Proof. Let $\Psi = (\beta, \psi)$ and $\Theta = (\gamma, \vartheta)$ and set $\mathcal{M}_i = (T_i, \mathfrak{A}_i)$, $\Phi_i = (\alpha_i, \varphi_i)$ and $\|\cdot\|_i := \varphi_i$. It is enough to show that $\|(\vartheta \circ_{\mathbf{K}} \psi)(a)\|_3 \le \|a\|_1$ for all $a \in |\mathfrak{A}_1|$. But this is straightforward since, by hypothesis, $\|\vartheta(\psi(a))\|_3 \le \|\psi(a)\|_2$ on the one hand, and $\|\psi(a)\|_2 \le \|a\|_1$ on the other, so that the conclusion follows from the transitivity of $\le$. ■

Define $\mathcal{C}_{\rm o}$ as the class of prenormed $\mathbf{K}$-models over $\mathcal{M}_0$ and $\mathcal{C}_{\rm h}$ as that of triples $(\mathsf{M}_1, \mathsf{M}_2, \Psi)$ such that M_i is a prenormed $\mathbf{K}$-model of $\mathfrak{T}$ over $\mathcal{M}_0$ and Ψ a $\mathbf{K}$-short morphism $\mathsf{M}_1 \to \mathsf{M}_2$. Then, take s and t to be the maps $\mathcal{C}_{\rm h} \to \mathcal{C}_{\rm o} : (\mathsf{M}_1, \mathsf{M}_2, \Phi) \mapsto \mathsf{M}_1$ and $\mathcal{C}_{\rm h} \to \mathcal{C}_{\rm o} : (\mathsf{M}_1, \mathsf{M}_2, \Phi) \mapsto \mathsf{M}_2$, respectively, and denote by i the function $\mathcal{C}_{\rm o} \to \mathcal{C}_{\rm h}$ sending a prenormed $\mathbf{K}$-model $\mathsf{M} = (\mathcal{M}, \Phi)$ to the triple $(\mathsf{M}, \mathsf{M}, \varepsilon)$, where ε is the identity $\mathcal{M} \to \mathcal{M}$ in $\mathbf{Prn_K}(\mathfrak{T})$. Lastly, let c be the partial function $\mathcal{C}_{\rm h} \times \mathcal{C}_{\rm h} \nrightarrow \mathcal{C}_{\rm h}$ specified as follows: Pick $\mathfrak{m} = (\mathsf{M}_1, \mathsf{M}_2, \Psi)$ and $\mathfrak{n} = (\mathsf{N}_1, \mathsf{N}_2, \Theta)$ in $\mathcal{C}_{\rm h}$. If $\mathsf{M}_2 \ne \mathsf{N}_1$, then $c(\mathfrak{m}, \mathfrak{n})$ is not defined. Otherwise, based on Lemma 4.1, assume $\Psi = (\beta, \psi)$ and $\Theta = (\gamma, \vartheta)$ and set $c(\mathfrak{m}, \mathfrak{n})$ equal to the triple $(\mathsf{M}_1, \mathsf{N}_2, \Pi)$, where $\Pi := (\gamma \circ_{\mathbf{Sgn}_1} \beta, \vartheta \circ_{\mathbf{K}} \psi)$.

It is easy to verify that the 6-uple $(\mathcal{C}_{\rm o}, \mathcal{C}_{\rm h}, s, t, i, c)$ gives a category. We call it the category of prenormed $\mathbf{K}$-models of $\mathfrak{T}$ over $\mathcal{M}_0$ and refer to its objects as prenormed $\mathbf{K}$-models of $\mathfrak{T}$ over $\mathcal{M}_0$. This category will be denoted, in general, by $\mathbf{Pnr_K}(\mathfrak{T}; \mathcal{M}_0)$, and especially by $\mathbf{Pnr_K}(T; \mathcal{M}_0)$ in the case where $\mathfrak{T} = \{T, T_0\}$ for some prealgebraic theory T (possibly equal to T_0). When $\mathcal{M}_0$ is a subnormed $\mathbf{K}$-model of $\mathfrak{T}$, then the objects in $\mathbf{Pnr_K}(\mathfrak{T}; \mathcal{M}_0)$ that are, indeed, subnormed $\mathbf{K}$-models of $\mathfrak{T}$ over $\mathcal{M}_0$, form a full subcategory of $\mathbf{Pnr_K}(\mathfrak{T}; \mathcal{M}_0)$. This is written as $\mathbf{Snr_K}(\mathfrak{T}; \mathcal{M}_0)$, or as $\mathbf{Snr_K}(T; \mathcal{M}_0)$ for $\mathfrak{T} = \{T, T_0\}$, and called the category of subnormed $\mathbf{K}$-models over $\mathcal{M}_0$.

A detailed study of the properties of these categories is beyond the scope of the present paper and we just restrict ourselves to a couple of considerations. The first is that $\mathbf{Snr_K}(\mathfrak{T}; \mathcal{M}_0)$ is a full subcategory of $\mathbf{Pnr_K}(\mathfrak{T}; \mathcal{M}_0)$ whenever $\mathcal{M}_0$ is a subnormed $\mathbf{K}$-model of $\mathfrak{T}$. As for the second, assume $\mathfrak{T} = \{T, T_0\}$ and $\mathfrak{T}_s = \{T_s, T_0\}$, where T and T_s are prealgebraic [resp. subalgebraic] theories and $T_s \le T$. Then, the "forgetful" functor $\mathcal{C}_{T_s} : \mathbf{Pnr_K}(\mathfrak{T}) \to \mathbf{Pnr_K}(\mathfrak{T}_s)$ [resp. $\mathcal{C}_{T_s}$:

$\mathbf{Snr_K}(\mathfrak{T}) \to \mathbf{Snr_K}(\mathfrak{T}_s)$] defined by the end of Section 2 gives rise to another "forgetful" functor $\mathcal{E}_{T_s} : \mathbf{Pnr_K}(T; \mathcal{M}_0) \to \mathbf{Pnr_K}(T_s; \mathcal{M}_0)$ [resp. $\mathcal{E}_{T_s} : \mathbf{Snr_K}(T; \mathcal{M}_0) \to \mathbf{Snr_K}(T_s; \mathcal{M}_0)$] by sending

(i) a prenormed [resp. subnormed] $\mathbf{K}$-model $(\mathcal{M}, \Phi)$ of T over $\mathcal{M}_0$ to $(\mathcal{C}_{T_s}(\mathcal{M}), \mathcal{C}_{T_s}(\Phi))$, the last being regarded as a prenormed [resp. subnormed] $\mathbf{K}$-model of T_s over $\mathcal{M}_0$;

(ii) a $\mathbf{K}$-short morphism $\Psi : (\mathcal{M}_1, \Phi_1) \to (\mathcal{M}_2, \Phi_2)$ of $\mathbf{Pnr_K}(T; \mathcal{M}_0)$ [resp. $\mathbf{Snr_K}(T; \mathcal{M}_0)$] to the $\mathbf{K}$-short morphism $(\mathcal{C}_{T_s}(\mathcal{M}_1), \mathcal{C}_{T_s}(\Phi_1)) \to (\mathcal{C}_{T_s}(\mathcal{M}_2), C_{T_s}(\Phi_2))$ of $\mathbf{Pnr_K}(T_s; \mathcal{M}_0)$ [resp. $\mathbf{Snr_K}(T_s; \mathcal{M}_0)$].

In particular, $\mathcal{E}_{T_s}$ returns a "forgetful" functor to $\mathbf{K}$ in the extreme case where T_s is the empty theory in the variables V (see Remark 11). It is then interesting to ask when $\mathcal{E}_{T_s}$ admits adjoints. However, the question, along with other properties of $\mathbf{Pnr_K}(T; \mathcal{M}_0)$ and $\mathbf{Snr_K}(T; \mathcal{M}_0)$, critically depends on the specificity of $\mathbf{K}$, T and T_s and will be investigated in future work.

5. Some worked examples

Unless explicitly overridden, the notation throughout is based on that of the previous section. Here, we show how the framework developed so far succeeds to capture essential features of the notion itself of norm as this is intended in the classical approach to the theory of normed groups, valuated rings and similar one-sorted structures (vector spaces and algebras will be discussed in Part II). In each of the examples examined, it is $\mathbf{K} = \mathbf{Set}$. Accordingly, any further reference to $\mathbf{K}$ is omitted and we use, e.g., "model" in place of "$\mathbf{K}$-model", "prenorm" [resp. "subnorm"] instead of "$\mathbf{K}$-prenorm" [resp. "$\mathbf{K}$-subnorm"], $\mathbf{Pnr}(\cdot)$ for $\mathbf{Pnr_K}(\cdot)$, and so on. We focus on a family $\mathfrak{T}$ consisting of two theories $T = (\sigma, \Xi)$ and $T_0 = (\sigma_0, \Xi_0)$, possibly equal to each other.

Remark 17. If $A \in \mathrm{obj}(\mathbf{Set})$, each global point of A can be identified with a standard element of A, and viceversa. Therefore, in what follows, we systematically interpret the membership symbol $\in$ in its standard set-theoretic sense rather than in generalized meaning of Remark 21.

That said, we take $\mathfrak{A} = (A, \sigma, \mathfrak{i})$ to be an algebraic model of T and $\mathfrak{A}_0 = (A_0, \sigma_0, \mathfrak{i}_0)$ a pivotal prealgebraic [resp. subalgebraic] model of T_0 (keep Remark 14 in mind). We set $\mathcal{M} := (T, \mathfrak{A})$ and $\mathcal{M}_0 := (T_0, \mathfrak{A}_0)$ and denote the pivot of $\mathfrak{A}_0$ by $\leq$. Furthermore, in the light of Remark 13, we assume that σ is a subsignature of σ_0 and concentrate only on $\mathcal{M}_0$-valued prenorms [resp. subnorms] on $\mathcal{M}$ of the form $\Phi = (\alpha, \|\cdot\|)$ such that α is the canonical injection $\sigma \to \sigma_0$, hence identifying Φ with $\|\cdot\|$ by a convenient abuse of notation. Lastly, for T_s a subtheory of T, we use $\mathcal{C}_{T_s}$ for the "forgetful" functor $\mathbf{Pnr}(T) \to \mathbf{Pnr}(T_s)$ defined by the end of Section 2.

Now, we pick a distinguished set of (non-logical) function symbols, $\Sigma_\mathrm{f} = \{+, \star, u, 0, 1\}$, and a distinguished set of (non-logical) relation symbols, $\Sigma_\mathrm{r} = \{\leq_+, \leq_\star, \leq_u, \leq_0, \leq_1\}$, and introduce a "reference signature" $\sigma_\mathrm{ref} = (\Sigma_\mathrm{f}, \Sigma_\mathrm{r}, \mathrm{ar}_\mathrm{ref})$, where ar_ref is defined in such a way that $+$, $\star$ and every member of Σ_r are binary, u is unary, and 0 and 1 are nullary. Then, as is usual, we call

(s.1) $\sigma_\mathrm{sgrp} := (+, \leq_+; \mathrm{ar}_\mathrm{sgrp})$ the signature of semigroups;
(s.2) $\sigma_\mathrm{mon} := (+, \leq_+; 0, \leq_0; \mathrm{ar}_\mathrm{mon})$ the signature of monoids;
(s.3) $\sigma_\mathrm{grp} := (+, \leq_+; u, \leq_u; 0, \leq_0; \mathrm{ar}_\mathrm{grp})$ the signature of groups;
(s.4) $\sigma_\mathrm{rg} := (+, \leq_+; \star, \leq_\star; 0, \leq_0; \mathrm{ar}_\mathrm{rg})$ the signature of semirings;
(s.5) $\sigma_\mathrm{rig} := (+, \leq_+; \star, \leq_\star; 0, \leq_0; 1, \leq_1; \mathrm{ar}_\mathrm{rig})$ the signature of unital semirings;
(s.6) $\sigma_\mathrm{rng} := (+, \leq_+; \star, \leq_\star; u, \leq_u; 0, \leq_0; \mathrm{ar}_\mathrm{rng})$ the signature of rings;
(s.7) $\sigma_\mathrm{ring} := (+, \leq_+; \star, \leq_\star; u, \leq_u; 0, \leq_0; 1, \leq_1; \mathrm{ar}_\mathrm{rng})$ the signature of unital rings.

Here, $\mathrm{ar}_\mathrm{sgrp}$, ar_mon, etc are the appropriate restrictions of ar_ref to $\{+, \leq_+\}$, $\{+, \leq_+, 0, \leq_0\}$, etc. We then say that a subnorm $\|\cdot\| : \mathcal{M} \to \mathcal{M}_0$, if any exists, is an $\mathcal{M}_0$-valued semigroup [resp.

group] subnorm (on $\mathcal{M}$) if T is the smallest subalgebraic theory of type $\sigma_{\rm sgrp}$ [resp. $\sigma_{\rm grp}$], an $\mathcal{M}_0$-valued monoid subnorm if T is the smallest subalgebraic theory of type $\sigma_{\rm mon}$, an $\mathcal{M}_0$-valued semiring [resp. ring] subnorm if T is the smallest subalgebraic theory of type $\sigma_{\rm rg}$ [resp. $\sigma_{\rm rng}$], and an $\mathcal{M}_0$-valued subnorm of unital semirings [resp. unital rings] if T is the smallest subalgebraic theory of type $\sigma_{\rm rig}$ [resp. $\sigma_{\rm ring}$] (cf. Remark 8). Thus, a subnorm $\|\cdot\| : \mathcal{M} \to \mathcal{M}_0$ is

(E.1) an $\mathcal{M}_0$-valued semigroup subnorm (on $\mathcal{M}$) if and only if

$$\|a+b\| \le_+ \|a\| + \|b\| \text{ for all } a, b \in A. \tag{9}$$

It is common that $\mathfrak{A}$ is a model of the subalgebraic theory $T_{\rm sgrp}$ of semigroups, which is the smallest subalgebraic theory of signature $\sigma_{\rm sgrp}$ including the axiom of associativity (see Remark 7) for the symbol $+$.

(E.2) an $\mathcal{M}_0$-valued monoid subnorm if and only if $\mathcal{C}_{\sharp_s T_{\rm sgrp}}(\|\cdot\|)$ is an $\mathcal{M}_0$-valued semigroup subnorm on $C_{\sharp_s T_{\rm sgrp}}(\mathcal{M})$ and $\|0\| \le_0 0$ (see Remark 8). Typically, $\mathfrak{A}$ models the subalgebraic theory $T_{\rm mon}$ of monoids, to wit, the smallest subalgebraic extension of $T_{\rm sgrp}$ comprising the axiom of neutrality (see Remark 7) for the symbol 0. When this happens, motivated by the "classical theory", we refer to an $\mathcal{M}_0$-valued monoid subnorm $\|\cdot\|$ (on $\mathcal{M}$) which is upward semidefinite [resp. definite] with respect to 0 as an $\mathcal{M}_0$-valued monoid seminorm [resp. norm], and call $\mathcal{M}$ a seminormed [resp. normed] monoid over $\mathcal{M}_0$. One usually takes $\mathcal{M}_0$ to be the additive monoid of the non-negative real numbers with its standard (order and algebraic) structure: then, Remark 15 *implies* $\|0\| = 0$.

(E.3) an $\mathcal{M}_0$-valued group subnorm if and only if

$$\|u(a)\| \le_u u(\|a\|) \text{ for all } a \in A \tag{10}$$

and $\mathcal{C}_{\sharp_s T_{\rm mon}}(\|\cdot\|)$ is an $\mathcal{M}_0$-valued monoid subnorm on $C_{\sharp_s T_{\rm mon}}(\mathcal{M})$. Commonly, $\mathfrak{A}$ is a model of the subalgebraic theory $T_{\rm grp}$ of groups, the smallest subalgebraic extension of $T_{\rm mon}$ containing the axiom of inverses (see Remark 7) for the triple $(+, u, 0)$. In these cases, u is usually represented by the symbol $-$, so the above Equation (10) reads as:

$$\|-a\| \le_u u(\|a\|) \text{ for all } a \in A \tag{11}$$

and indeed as: $\|-a\| \le -\|a\|$ for all $a \in A$ if $T_{\rm grp} \le T_0$. Therefore, one concludes that group subnorms are, in some sense, "naturally negative" as far as we look at them as morphisms between structures of the very same type, i.e., groups. While intriguing, this is not completely satisfactory, for the relevant case of standard (positive definite) group norms [2, p. 5] is not covered. However, similar structures can be brought within the scope of our framework in the light of one trivial consideration: That the target of a standard group norm is taken to be $\mathbb{R}_0^+$, which is everything but a group. With this in mind, the most obvious workaround is to assume that T_0 is not a supertheory of T, but instead the smallest subalgebraic $\sigma_{\rm grp}$-theory. Then, u can be interpreted as the identity map on A_0 and Equation (11) becomes: $\|-a\| \le_u \|a\|$ for all $a \in A$. If $\le$ is a partial order, it follows from here that $\|\cdot\|$ is *necessarily* symmetric. This is another byproduct of our approach. It suggests that "asymmetric group norms" (cf. [2, Remark 2]) do not really exist as such: They can, e.g., as monoid norms but not as group norms, which is absolutely reasonable if we think of the fact that an "asymmetric group norm" is ultimately defined without any specific requirement about inverses. Starting from these considerations, we then refer to an $\mathcal{M}_0$-valued group subnorm $\|\cdot\|$ (on $\mathcal{M}$), which is upward semidefinite [resp. definite] with respect to 0, as an $\mathcal{M}_0$-valued group seminorm [resp. norm] (on $\mathcal{M}$), and call $\mathcal{M}$ a seminormed [resp. normed] group over $\mathcal{M}_0$.

(E.4) an $\mathcal{M}_0$-valued semiring subnorm if and only if

$$\|a \star b\| \leq_\star \|a\| \star \|b\| \text{ for all } a, b \in A \tag{12}$$

and $\mathcal{C}_{\sharp_s T_{\rm mon}}(\|\cdot\|)$ is an $\mathcal{M}_0$-valued monoid subnorm on $C_{\sharp_s T_{\rm mon}}(\mathcal{M})$. Note how this suggests that norms on ring-like structures are "inherently submultiplicative": "Multiplicativeness" is covered by assuming that $\leq_\star$ is (interpreted as) the equality relation on A_0, and the same applies to different operations, to the extent that, from an abstract point of view, there is no apparent reason to focus on the one rather than the others. Motivated by the terminology of the theory of valuated rings, we then refer to an $\mathcal{M}_0$-valued semiring subnorm $\|\cdot\|$ (on $\mathcal{M}$), which is upward semidefinite [resp. definite] with respect to 0 and "multiplicative" with respect to $\star$, as an $\mathcal{M}_0$-valued semiring semivaluation [resp. valuation] (on $\mathcal{M}$), and call $\mathcal{M}$ a semivaluated [resp. valuated] semiring over $\mathcal{M}_0$. In most applications, $\mathfrak{A}$ and $\mathfrak{A}_0$ will be models of the subalgebraic theory $T_{\rm rg}$ of semirings, i.e., the smallest subalgebraic extension of $T_{\rm mon}$ containing the axioms of left and right distributiveness of $\star$ over $+$ and the axiom of associativity for $\star$. E.g., this is the case with the semiring of non-negative real numbers (with the usual structure inherited from the real field).

(E.5) an $\mathcal{M}_0$-valued ring subnorm if and only if $\mathcal{C}_{\sharp_s T_{\rm grp}}(\|\cdot\|)$ is an $\mathcal{M}_0$-valued group subnorm on $\mathcal{C}_{\sharp_s T_{\rm grp}}(\mathcal{M})$ and $\mathcal{C}_{\sharp_s T_{\rm rg}}(\|\cdot\|)$ is an $\mathcal{M}_0$-valued group semiring subnorm on $\mathcal{C}_{\sharp_s T_{\rm rg}}(\mathcal{M})$. Thus, all the considerations previously made on group and semiring subnorms also apply to ring subnorms. In particular, we refer to an $\mathcal{M}_0$-valued ring subnorm $\|\cdot\|$ (on $\mathcal{M}$), which is upward semidefinite [resp. definite] with respect to 0 and "multiplicative" with respect to $\star$, as an $\mathcal{M}_0$-valued ring semivaluation [resp. valuation] (on $\mathcal{M}$), and then call $\mathcal{M}$ a semivaluated [resp. valuated] ring over $\mathcal{M}_0$. In common cases, $\mathfrak{A}$ is a model of the subalgebraic theory $T_{\rm rng}$ of rings, i.e., the smallest subalgebraic theory containing both the axioms of $T_{\rm grp}$ and those of $T_{\rm rg}$.

(E.6) an $\mathcal{M}_0$-valued subnorm of unital semirings if and only if $\mathcal{C}_{\sharp_s T_{\rm rg}}(\|\cdot\|)$ is an $\mathcal{M}_0$-valued semiring subnorm on $\mathcal{C}_{\sharp_s T_{\rm rg}}(\mathcal{M})$ and $\|1\| \leq_1 1$. The same considerations previously made on the symbol 0 in the case of group subnorms apply to 1. Furthermore, mimicking the case of semiring subnorms, we refer to an $\mathcal{M}_0$-valued subnorm $\|\cdot\|$ of unital semirings (on $\mathcal{M}$), which is upward semidefinite [resp. definite] with respect to 0 and "multiplicative" with respect to $\star$, as an $\mathcal{M}_0$-valued semivaluation [resp. valuation] of unital semirings (on $\mathcal{M}$), and then call $\mathcal{M}$ a semivaluated [resp. valuated] unital semiring over $\mathcal{M}_0$. If $\|\cdot\|$ is a $\mathcal{M}_0$-valued semivaluation of unital semirings (on $\mathcal{M}$) for which $\|1\|$ is a unit in $\mathcal{M}_0$, i.e., an invertible element with respect to $\star$, and $\leq_\star$ is compatible with $\star$, in the sense that $a_1 \star a_2 \leq_\star b_1 \star b_2$ for $a_i, b_i \in A$ and $a_i \leq_\star b_i$, then $\|1\| = \|1 \star 1\| \leq_\star \|1\| \star \|1\|$ *implies* $\|1\| = 1$, and hence $\|1\| = 1$ (one more unexpected outcome of our approach). In relevant applications, $\mathfrak{A}$ will model the subalgebraic theory $T_{\rm rig}$ of unital semirings, i.e., the smallest subalgebraic extension of $T_{\rm rg}$ including the axiom of neutrality for 1, as for the non-negative real numbers with their usual order and algebraic structure.

(E.7) an $\mathcal{M}_0$-valued subnorm of unital rings if and only if $\mathcal{C}_{\sharp_s T_{\rm rig}}(\|\cdot\|)$ is an $\mathcal{M}_0$-valued subnorm of unital semirings on $\mathcal{C}_{\sharp_s T_{\rm rig}}(\mathcal{M})$ and $\mathcal{C}_{\sharp_s T_{\rm grp}}(\|\cdot\|)$ is an $\mathcal{M}_0$-valued group subnorm on $\mathcal{C}_{\sharp_s T_{\rm grp}}(\mathcal{M})$. The very same considerations previously made in the case of group subnorms and subnorms of unital semirings apply to ring subnorms. We refer to an $\mathcal{M}_0$-valued subnorm $\|\cdot\|$ of unital rings (on $\mathcal{M}$), which is upward semidefinite [resp. definite] with respect to 0 and "multiplicative" with respect to $\star$, as an $\mathcal{M}_0$-valued

semivaluation [resp. valuation] of unital rings (on $\mathcal{M}$), and then call $\mathcal{M}$ a semivaluated [resp. valuated] unital ring over $\mathcal{M}_0$. Typically, $\mathfrak{A}$ models the subalgebraic theory $T_{\rm ring}$ of unital rings, the smallest subalgebraic extension of $T_{\rm rig}$ including the axiom of inverses for the triple $(+, u, 0)$.

$\mathcal{M}_0$-valued semigroup subnorms, monoid subnorms, etc are defined and characterized in the very same way, by replacing "subalgebraic" with "prealgebraic" and $\sharp_s$ with $\sharp_p$ in all of their occurrences in the above discussion. Furthermore, most of the considerations made in the subalgebraic case still hold in the prealgebraic one, except for those based on Remark 15.

Field valuations and norms of vector spaces over a fixed valuated field, together with variants thereof, will be discussed in Part II as special instances of many-sorted subnormed structures.

Appendix A. A resumé of the very basics of category theory

Following [16], we define a category as a 6-uple $(\mathcal{C}_{\rm o}, \mathcal{C}_{\rm h}, s, t, i, c)$, where $\mathcal{C}_{\rm o}$ and $\mathcal{C}_{\rm h}$ are classes, the former referred to as the collection of objects, the latter as the collection of morphisms or arrows; s and t are functions $\mathcal{C}_{\rm h} \to \mathcal{C}_{\rm o}$ which assign, to every arrow, its source and target; i is a further function $\mathcal{C}_{\rm o} \to \mathcal{C}_{\rm h}$ sending each object A to a distinguished morphism, called the (local) identity on A; c is a partial operation $\mathcal{C}_{\rm h} \times \mathcal{C}_{\rm h} \nrightarrow \mathcal{C}_{\rm h}$ (called composition) whose domain is the class of all pairs $((f, g_1), (g_2, h)) \in \mathcal{C}_{\rm h} \times \mathcal{C}_{\rm h}$ with $g_1 = g_2$ (see Subsection 1.1 for a formal definition of partial maps between classes); and all is accompanied by the usual axioms that s, t, id and $\circ$ are required to satisfy, to wit,

(i) $s(c(f, g)) = s(f)$ and $t(c(f, g)) = t(g)$ for every $(f, g) \in \mathrm{dom}(c)$.
(ii) $s(i(A)) = t(i(A)) = A$.
(iii) $c(f, c(g, h)) = c(c(f, g), h)$ for all $f, g, h \in \mathcal{C}_{\rm h}$ such that $(f, g), (g, h) \in \mathrm{dom}(c)$.
(iv) $c(f, i(t(f))) = c(i(s(f)), f) = f$ for each $f \in \mathcal{C}_{\rm h}$.

Property (iii) is spelled by saying that c is associative. If $\mathbf{C}$ is a category, one denotes $\mathrm{obj}(\mathbf{C})$ the class of its objects and $\mathrm{hom}(\mathbf{C})$ the one of its arrows. We write $\mathrm{src}_{\mathbf{C}}$ and $\mathrm{trg}_{\mathbf{C}}$ for the functions mapping a morphism to its source and target, respectively, and $\mathrm{id}_{\mathbf{C}}$ for that sending an object A to the identity on A. For $A, B \in \mathrm{obj}(\mathbf{C})$, we adopt the notation $(f : A \to B) \in \mathbf{C}$ to mean that $f \in \mathrm{hom}(\mathbf{C})$, $\mathrm{src}_{\mathbf{C}}(f) = A$ and $\mathrm{trg}_{\mathbf{C}}(f) = B$. This simplifies to the usual $f : A \to B$, or one says that f is an arrow $A \to B$, when $\mathbf{C}$ is clear from the context. We use $\circ_{\mathbf{C}}$ for the composition law of $\mathbf{C}$ and $\mathrm{hom}_{\mathbf{C}}(A, B)$ for the collection of arrows $(f : A \to B) \in \mathbf{C}$, or simply $\mathrm{hom}_{\mathbf{C}}(A)$ when $A = B$. Lastly, for $(f, g) \in \mathrm{dom}(\circ_{\mathbf{C}})$, we write $g \circ_{\mathbf{C}} f$ in place of $\circ_{\mathbf{C}}(f, g)$ and refer to $g \circ_{\mathbf{C}} f$, as is customary, as the composition of g with f (cf. [3, Section 1.2]).

Remark 18. We denote $\cong_{\mathbf{C}}$ the equivalence on $\mathrm{obj}(\mathbf{C})$ defined by: $A \cong_{\mathbf{C}} B$ for $A, B \in \mathrm{obj}(\mathbf{C})$ if and only if there is an isomorphism $u : A \to B$. Then, for $A \in \mathrm{obj}(\mathbf{C})$, we indicate by $\mathrm{iso}_{\mathbf{C}}(A)$ the equivalence class of A in the quotient of $\mathrm{obj}(\mathbf{C})$ by $\cong_{\mathbf{C}}$ and refer to it as the isomorphism class of A in $\mathbf{C}$. If $B, C \in \mathrm{iso}_{\mathbf{C}}(A)$, one says that B and C are isomorphic (cf. [3, Section 1.9]).

Let $\mathbf{C}$ be a category and $\{A_i\}_{i\in I}$ an indexed set of objects of $\mathbf{C}$. We write $\prod_{\mathbf{C}}\{A_i\}_{i\in I}$, whenever it exists, for the $\mathbf{C}$-product of the A_i's, i.e., for the class of all pairs $(P, \{\pi_i\}_{i\in I})$ such that $P \in \mathrm{obj}(\mathbf{C})$, $\pi_i \in \mathrm{hom}_{\mathbf{C}}(P, A_i)$ for each $i \in I$ and the following universal property is satisfied: If $(Q, \{\omega_i\}_{i\in I})$ is any other pair with $Q \in \mathrm{obj}(\mathbf{C})$ and $\omega_i \in \mathrm{hom}_{\mathbf{C}}(Q, A_i)$, there exists a unique morphism $u : Q \to P$ such that $\omega_i = \pi_i \circ_{\mathbf{C}} u$ for all $i \in I$. The universal arrow u is denoted by $\prod_{\mathbf{C}}\langle\omega_i, \pi_i\rangle_{i\in I}$, or equivalently by $\langle\omega_1, \pi_1\rangle \times_{\mathbf{C}} \langle\omega_2, \pi_2\rangle \times_{\mathbf{C}} \cdots \times_{\mathbf{C}} \langle\omega_n, \pi_n\rangle$ if I is finite and $n := |I|$. One refers to P as a product object of the A_i's, to the π_i's as (canonical) projections

from P, and to $\mathbf{iso}_\mathbf{C}(P)$ as the isomorphism class of $\prod_\mathbf{C}\{A_i\}_{i\in I}$ (cf. [3, Section 2.1]). If I is empty, P is interpreted as a terminal object of the category [3, Section 2.3].

Remark 19. A fixed object in the isomorphism class of the product of the A_i's can have infinitely many different classes of projections associated with it [3, Example 2.1.7.i]. Therefore, to avoid ambiguity, one should always specify which projections must be considered in a certain context. Nevertheless, when there is no need to make explicit reference to any particular set of projections or these are clear from the context, one writes $P \in \prod_\mathbf{C}\{A_i\}_{i\in I}$ to mean that P is an object of $\mathbf{C}$ in a product pair $(P, \Pi) \in \prod_\mathbf{C}\{A_i\}_{i\in I}$.

Remark 20. If there is no likelihood of confusion, any explicit reference to $\mathbf{C}$ is dropped in the above notations or in others similar and one uses, e.g., $\circ$ in place of $\circ_\mathbf{C}$, $\cong$ for $\cong_\mathbf{C}$, and so on. In particular, one writes $\prod_{i\in I} A_i$ instead of $\prod_\mathbf{C}\{A_i\}_{i\in I}$. Moreover, $\prod_{i\in I} A_i$ is denoted by $A_1 \times A_2 \times \cdots \times A_n$ if I is finite and $n := |I|$, and indeed by A^n if the A_i's are all equal to one same object A.

Definition 5. Let $\{A_i\}_{i\in I}$ be an indexed family of $\mathrm{obj}(\mathbf{C})$ and suppose that the A_i's have a product. A relation on $\{A_i\}_{i\in I}$ (in $\mathbf{C}$) is then any pair (r, Π) for which $(r : R \to P) \in \mathbf{C}$ is a monomorphism [3, Section 1.7] and $(P, \Pi) \in \prod_\mathbf{C}\{A_i\}_{i\in I}$. This is called an n-ary relation if I is finite and $n := |I|$, and indeed an n-ary relation on A if the A_i's are all equal to one object A.

Remark 21. If $\mathbf{C}$ has a terminal object $\top$ and $A \in \mathrm{obj}(\mathbf{C})$, one refers to an arrow $a \in \mathrm{hom}(\mathbf{C})$ with $\mathrm{src}(a) \cong \top$ as a [global] point of A (cf. [5, Definition 5.1.4]) and defines a [global] point of $\mathbf{C}$ as any morphism $a \in \mathrm{hom}(\mathbf{C})$ such that $\mathrm{src}(a) \cong \top$: note that a point is always a monomorphism [3, Definition 1.7.1]. We call $\mathrm{src}(a)$ the origin of a and write $a \in A$ to mean that a is a point of A. With this in hand, pick a nonempty finite subcollection $\{A_i\}_{i=1}^n$ of $\mathrm{obj}(\mathbf{C})$ and a family $\{a_i\}_{i=1}^n$ of *parallel* points of $\mathbf{C}$ with $a_i \in A_i$. If $(P, \{\pi_i\}_{i=1}^n) \in \prod_{i=1}^n A_i$, $\varphi \in \mathrm{hom}(P)$ and $\varrho = (r, \{\pi_i\}_{i=1}^n)$ is a relation on $\{A_i\}_{i=1}^n$, we then use

(i) $(a_1, a_2, \ldots, a_n)_\Pi$ for the morphism $\prod_{i=1}^n \langle a_i, \pi_i\rangle$;
(ii) $\varphi(a_1, a_2, \ldots, a_n)_\Pi$ in place of $\varphi \circ \prod_{i=1}^n \langle a_i, \pi_i\rangle$;
(iii) $(a_1, a_2, \ldots, a_n) \in \varrho$ if there is a point $\mu : \bullet \to \mathrm{src}(r)$ such that $\prod_{i=1}^n \langle a_i, \pi_i\rangle = r \circ \mu$,

where $\bullet$ is the common origin of the a_i's and $\Pi := \{\pi_i\}_{i=1}^n$. Observe that $\varphi(a_1, a_2, \ldots, a_n)_\Pi$ is still a point of $\mathbf{C}$, indeed parallel to the a_i's (this is a trivial but fundamental property for the consistency of our approach). The notation is suggestive of the fact that a point can be regarded as a generalization of the set-theoretic notion of element. Any explicit reference to Π is omitted from it, to the degree of using ϱ in place of r, whenever the π_i's are clear from the context.

Definition 6. Assume $\mathbf{C}$ is a category with binary products and a terminal object $\top$, and hence with all finite products [3, p. 40]. Pick $A \in \mathrm{obj}(\mathbf{C})$ and let ϱ be a binary relation on A. We say that ϱ is reflexive if $(a, a) \in \varrho$ for every $a \in A$; antisymmetric if, for all parallel $a, b \in A$, it holds that $(a, b), (b, a) \in \varrho$ only if $a = b$; transitive if, for all parallel $a, b, c \in A$, one has that $(a, c) \in \varrho$ whenever $(a, b), (b, c) \in \varrho$. Then, ϱ is called a preorder (on the object A) if it is reflexive and transitive, and a partial order if it is an antisymmetric preorder: In the former case, we refer to (A, ϱ) as a preordered object of $\mathbf{C}$; in the latter, as a partially ordered object, or pod.

These definitions are different from those analogously given for internal relations, especially in reference to the notion of congruence [17]. Yet, they are more than suitable for our purposes.

Examples. Having fixed the bulk of the terminology and notation used here and in future work to deal with categories, we conclude this appendix by recalling the definition of a few basic categories that will be considered at several points in our work (cf. [1, Section 1.4]):

(c.1) **Rel**, the category having sets as objects and all triples $\mathfrak{R}$ of type (X, Y, R) as morphisms, where X and Y are sets and R is a subset of the Cartesian product $X \times Y$. When this does not lead to confusion, we identify $\mathfrak{R}$ with R. The composition of two relations $R : X \to Y$ and $S : Y \to Z$ is defined by the triple (X, Z, T), where $T \subseteq X \times Z$ and $(x, z) \in T$ for $x \in X$ and $z \in Z$ if and only if $(x, y) \in R$ and $(y, z) \in S$ for some $y \in Y$.

(c.2) **Set**, the wide subcategory of **Rel** whose morphisms are functions (a subcategory $\mathbf{S}$ of a category $\mathbf{C}$ is said to be wide if $\mathrm{obj}(\mathbf{S}) = \mathrm{obj}(\mathbf{C})$).

(c.3) $\mathbf{Pre}(\mathbf{C})$, the category having as objects the preordered objects of a given category $\mathbf{C}$ with finite products and as morphisms all triples $\mathfrak{f}$ of the form $(\mathcal{P}, \mathcal{Q}, f)$ for which $\mathcal{P} = (P, \varrho)$ and $\mathcal{Q} = (Q, \eta)$ are preordered objects of $\mathbf{C}$ and $f \in \hom_{\mathbf{C}}(P, Q)$ is such that $(f(x), f(y)) \in \eta$ for all parallel points $x, y \in P$ with $(x, y) \in \varrho$. We call f a monotonic arrow $\mathcal{P} \to \mathcal{Q}$ and use f as a shorthand of $\mathfrak{f}$ when there is no danger of confusion. The composition of two morphisms $(\mathcal{P}, \mathcal{Q}, f)$ and $(\mathcal{Q}, \mathcal{R}, g)$ is $(\mathcal{P}, \mathcal{R}, g \circ_{\mathbf{C}} f)$.

(c.4) $\mathbf{Pos}(\mathbf{C})$, the full subcategory of $\mathbf{Pre}(\mathbf{C})$ whose objects are pods of $\mathbf{C}$.

The local identities and the maps of source and target are specified in the most obvious way.

This completes our brief introduction to the categorical language. For anything else not explicitly mentioned here, the interested reader will be referred to the three volumes of Borceux's precious monograph [3]-[5].

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